Probability Generating Functions 1 MS

Q1.

6(a)	Probabilities $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ for 0, 1, 2, 3 reds	1	**B1**	
	$G_X(t) = \frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3$	1	**B1FT**	Follow through their probabilities so long as $\Sigma p = 1$
		2		
6(b)	Correct value $\frac{9}{20}$ or 0.45 for $P(Y = 1)$ or $P(Y = 2)$	1	**B1**	
	Probabilities $\frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}$ for 0, 1, 2, 3 reds	1	**B1**	
	$G_Y(t) = \frac{1}{20} + \frac{9}{20}t + \frac{9}{20}t^2 + \frac{1}{20}t^3$	1	**B1FT**	Follow through their probabilities so long as $\Sigma p = 1$
		3		
6(c)	$G_Z(t) = G_X(t) \times G_Y(t)$	1	**B1**	Stated or implied
	Complete expansion of the product of their cubics	1	**M1**	
	$G_Z(t) = \frac{1}{160}(1 + 12t + 39t^2 + 56t^3 + 39t^4 + 12t^5 + t^6)$	1	**A1**	Or equivalent, e.g. with explicit fractional coefficients
		3		
6(d)	Attempt to differentiate $G_Z(t)$ and evaluate $G'_Z(1)$	1	**M1**	
	$E(Z) = 3$	1	**A1**	Correctly obtained from $G'_Z(1)$
	Attempt to find second derivative $G''_Z(t)$	1	**M1**	
	Use of $G''_Z(1) + G'_Z(1) - (G'_Z(1))^2$	1	**M1**	
	$\text{Var}(Z) = 1.2$	1	**A1**	Correct value correctly obtained
		5		

Q2.

6(a)	$P(BB) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$ $P(RB, BR) = \frac{6}{10} \times \frac{4}{9} = \frac{8}{15}$ $P(RR) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$	**M1**
	$\frac{1}{3} + \frac{8}{15}t + \frac{2}{15}t^2$ (**FT** *their* probabilities)	**A1 FT**
		2
6(b)	$P(1\text{ H}) = \frac{2}{3}(1-p) + \frac{1}{3}p = \frac{7}{12}$	**M1**
	$p = \frac{1}{4}$	**A1**
	PGF of $Y = \frac{1}{4} + \frac{7}{12}t + \frac{1}{6}t^2$	**A1**
		3
6(c)	PGF of $Z = \left(\frac{1}{3} + \frac{8}{15}t + \frac{2}{15}t^2\right)\left(\frac{1}{4} + \frac{7}{12}t + \frac{1}{6}t^2\right)$	**B1M1**
	$= \frac{1}{180}\left(15 + 59t + 72t^2 + 30t^3 + 4t^4\right)$ **AEF**	**A1**
		3
6(d)	Attempt to differentiate their $G_Z(t)$ and evaluate $G'_Z(1)$	**M1**
	$E(Z) = (59 + 144 + 90 + 16)/180 = \frac{309}{180} = 1.72$	**A1**
		2

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Q3.

4(a)	$G_X(t) = 0.2t + 0.5t^2 + 0.3t^3$	**B1**
	$G_Y(t) = \left(0.2t + 0.5t^2 + 0.3t^3\right)^2$	**M1**
	$= 0.04t^2 + 0.2t^3 + 0.37t^4 + 0.3t^5 + 0.09t^6$	**A1**
		3
4(b)	$G'_Y(t) = 0.08t + 0.6t^2 + 1.48t^3 + 1.5t^4 + 0.54t^5$	**M1**
	$E(Y) = 4.2$	**A1**
	$G''_Y(t) = 0.08 + 1.2t + 4.44t^2 + 6t^3 + 2.7t^4$	**M1**
	Use $G''_Y(1) + G'_Y(1) - (G'_Y(1))^2$	**M1**
	$\text{Var}(Y) = 14.42 + 4.2 - 4.2^2 = 0.98$	**A1**
		5

Q4.

5(a)	$G_X(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$	**B1**	Accept $(0.5+0.5t)^2$
		1	
5(b)	$P(0H) = \frac{12}{36}$ $P(1H) = \frac{16}{36}$ $P(2H) = \frac{7}{36}$ $P(3H) = \frac{1}{36}$	**M1 A1**	Attempt at probs, at least 2 correct All correct
	$G_Y(t) = \frac{12}{36} + \frac{16}{36}t + \frac{7}{36}t^2 + \frac{1}{36}t^3$	**B1 FT**	FT *their* probabilities, must be cubic with 4 non-zero terms
		3	
5(c)	$G_Z(t) = \left(\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2\right)\left(\frac{12}{36} + \frac{16}{36}t + \frac{7}{36}t^2 + \frac{1}{36}t^3\right)$	**M1**	Attempt to multiply their two PGF
	$= \frac{1}{144}\left(12 + 40t + 51t^2 + 31t^3 + 9t^4 + t^5\right)$	**M1 A1**	Obtain quintic expression and collect terms
		3	
5(d)	$G'_Z(t) = \frac{1}{144}\left(40 + 102t + 93t^2 + 36t^3 + 5t^4\right)$	**M1**	Differentiate
	$E(Z) = G'_Z(1) = \frac{23}{12} = (= 1.92)$	**A1**	
		2	
5(e)	2	**B1 FT**	FT power of term with largest coefficient in *their* $G_Z(t)$
		1	

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Q5.

5(a)	P($X = r$) = $^nC_r p^r (1-p)^{n-r}$ or $^nC_r p^r q^{n-r}$	B1	
	$G_X(t) = \sum_0^n nCr\, p^r (1-p)^{n-r} t^r$	M1	Accept minimum of 4 terms including the last. Shown with specific value of n is M0
	$\sum_0^n nCr(pt)^r (1-p)^{n-r}$ $= (q+pt)^n$	A1	At least one intermediate step to be shown, with p and t grouped AG
		3	
5(b)	$G'_X(t) = n(q+pt)^{n-1} \times p$	M1	
	So $E(X) = G'_X(1) = np(q+p)^{n-1}$ and $q+p=1$ so $E(X) = np$	A1	
	$G''_X(t) = n(n-1)(q+pt)^{n-2} \times p \times p$	M1	
	$\text{Var}(X) = n(n-1)p^2 + np - (np)^2$	M1	
	$np(1-p)$	A1	
		5	

Q6.

6(a)	P(3R, 2R, 1R, 0R) = $\frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0$	M1	At least 2 correct probabilities used in a polynomial.
	$G_X(t) = \frac{1}{5}t + \frac{3}{5}t^2 + \frac{1}{5}t^3$	A1	
		2	
6(b)	P(2H, 1H, 0H) = $\frac{1}{16}, \frac{6}{16}, \frac{9}{16}$	M1	One correct probability and all three adding to 1 used in a polynomial.
	$G_Y(t) = \frac{9}{16} + \frac{6}{16}t + \frac{1}{16}t^2$	A1	
		2	
6(c)	Attempt to multiply results from part **(a)** and part **(b)**	M1	
	Obtain polynomial	M1	
	$\frac{9}{80}t + \frac{33}{80}t^2 + \frac{28}{80}t^3 + \frac{9}{80}t^4 + \frac{1}{80}t^5$	A1	Accept exact equivalent decimals.
		3	
6(d)	$E(Z) = G_Z'(1) = \frac{1}{80}(9+66+84+36+5)$	M1	Differentiate and put $t = 1$.
	$= \frac{200}{80} = 2.5$	A1	
	$G_Z''(1) = \frac{1}{80}(66+168+108+20) = 4.525 = \left[\frac{362}{80}\right]$	M1	

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6(d)	$\text{Var}(Z) = G''(1) + G'(1) - (G'(1))^2$	**M1**	Use result.
	$\frac{362}{80} + \frac{200}{80} - \left(\frac{200}{80}\right)^2 = 0.775 = \left[\frac{31}{40}\right]$	**A1**	
		5	