# Proof by Induction 1

Q1. Prove by induction that

$$\sum_{r=1}^{n}(3r^5 + r^3) = \tfrac{1}{2}n^3(n+1)^3,$$

for all $n \geqslant 1$. [5]

Use this result together with the List of Formulae (MF10) to prove that

$$\sum_{r=1}^{n} r^5 = \tfrac{1}{12}n^2(n+1)^2\mathrm{Q}(n),$$

where $\mathrm{Q}(n)$ is a quadratic function of $n$ which is to be determined. [3]

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Q2. The sequence $x_1, x_2, x_3, \ldots$ is such that $x_1 = 3$ and

$$x_{n+1} = \frac{2x_n^2 + 4x_n - 2}{2x_n + 3}$$

for $n = 1, 2, 3, \ldots$. Prove by induction that $x_n > 2$ for all $n$. [6]

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Q3. Prove by mathematical induction that, for all non-negative integers $n$, $7^{2n+1} + 5^{n+3}$ is divisible by 44. [5]

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Q4. It is given that $\mathrm{f}(n) = 3^{3n} + 6^{n-1}$.

**(i)** Show that $\mathrm{f}(n+1) + \mathrm{f}(n) = 28(3^{3n}) + 7(6^{n-1})$. [2]

**(ii)** Hence, or otherwise, prove by mathematical induction that $\mathrm{f}(n)$ is divisible by 7 for every positive integer $n$. [4]

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Q5. Let $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Prove by mathematical induction that, for every positive integer $n$,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}.$$ [5]

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Q6. Prove by mathematical induction that, for all positive integers $n$,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = 2^{\frac{1}{2}n}\mathrm{e}^x \sin\left(x + \tfrac{1}{4}n\pi\right).$$ [7]

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Q7. Prove by mathematical induction that, for all positive integers $n$,

$$\frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right) = (-1)^n \frac{n!2^n}{(2x+3)^{n+1}}. \qquad [6]$$

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Q8. Let $S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{N}{(N+1)!}$. Prove by mathematical induction that, for all positive integers $N$,

$$S_N = 1 - \frac{1}{(N+1)!}. \qquad [5]$$

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Q9. It is given that $\phi(n) = 5^n(4n+1) - 1$, for $n = 1, 2, 3, \ldots$. Prove, by mathematical induction, that $\phi(n)$ is divisible by 8, for every positive integer $n$. [7]

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