Proof by Induction 1 - MS

Q1.	Verifies H_1 to be true	B1
	$H_k : \sum_{r=1}^{k} (3r^5 + r^3) = (1/2)k^3(k+1)^3$	B1
	$H_k \Rightarrow \sum_{r=1}^{k+1} (3r^5 + r^3) = (1/2)k^3(k+1)^3 + 3(k+1)^5 + (k+1)^3$	M1
	$= \ldots = (1/2)(k+1)^3(k+2)^3$	A1
	Thus $H_k \Rightarrow H_{k+1}$ and concludes	A1
	$3\sum_{r=1}^{n} r^5 + (1/4)n^2(n+1)^2 = (1/2)n^3(n+1)^3$	M1
	$\Rightarrow \ldots \Rightarrow \sum_{r=1}^{n} r^5 = (1/12)n^2(n+1)^2(2n^2 + 2n - 1)$	M1A1

Q2.	$H_k : x_k > 2$ for some k	B1
	$x_{k+1} - 2 = (2x_k^2 - 8)/(2x_k + 3)$	M1A1
	$H_k \Rightarrow 2x_k^2 - 8 > 0 \Rightarrow x_{k+1} > 2 \Rightarrow H_{k+1}$	A1
	$x_1 = 3 > 2 \Rightarrow H_1$ is true	B1 CWO
	Completion of the induction argument	A1 [6]

Alternatively for lines 2 and 3:

$x_{k+1} = x_k + \frac{1}{2} - \frac{3\frac{1}{2}}{(2x_k + 3)}$	M1A1
$H_k \Rightarrow 2x_k + 3 > 7 \Rightarrow H_{k+1}$	A1
OR $x_{k+1} = x_k + \frac{x_{k-2}}{(2x_k + 3)}$	M1A1
$x_k > 2 \Rightarrow x_{k+1} > 2$	A1
OR $x_{k+1} - x_k = \frac{x_{k-2}}{(2x_k + 3)}$	M1A1
$x_k > 2 \Rightarrow x_{k+1} > x_k > 2$	A1

Minimum conclusion is 'Hence true for $n \geqslant 1$'.

Q3. $n = 0$: $7^1 + 5^3 = 132$ which is divisible by 44 — B1

Assume $7^{2k+1} + 5^{k+3}$ is divisible by 44 — B1

Consider $7^{2(k+1)+1} + 5^{(k+1)+3} = 7^2 7^{2k+1} + 5.5^{k+3}$ — M1 — $(k + 1)$ th term

$= 49(7^{2k+1} + 5^{k+3}) - 44.5^{k+3}$ — M1 — in appropriate form

which is divisible by 44 — A1 — convincing argument — [5]

Alternative solution for final three marks:

Consider $(7^{2k+3} + 5^{k+4}) - (7^{2k+1} + 5^{k+3})$ — M1

$= 48(7^{2k+1} + 5^{k+3}) - 44.5^{k+3}$ — M1 — in appropriate form

which is divisible by 44 — A1 — convincing argument

Q4.

4 (i)	Establishes initial result.	$f(n)+f(n+1)=3^{3n}+6^{n-1}+3^{3n+3}+6^n$ $=3^{3n}(1+27)+6^{n-1}(1+6)$ $=28(3^{3n})+7(6^{n-1})$ (AG)	M1 A1	2	
(ii)	States inductive hypothesis. Proves base case. Shows $P_k \Rightarrow P_{k+1}$. States conclusion.	H_k: $f(k) = 7\lambda$ $3^3+6^0=28=4\times 7 \Rightarrow H_1$ is true $f(k+1)+f(k) = f(k+1)+7\lambda = 28(3^{3k})+7(6^{k-1})$ $= 7\mu$ $\Rightarrow f(k+1)=7(\mu-\lambda)\ \therefore H_k \Rightarrow H_{k+1}$ (Hence by the principle of mathematical induction H_n is) **true for all positive integers n.**	B1 B1 M1 A1	4	**[6]**

Q5.

	States proposition.	Let P_n be the proposition: $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \Rightarrow \mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n-1) \\ 0 & 1 \end{pmatrix}$			
	Shows base case is true.	$\mathbf{A}^1 = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 3\times(2-1) \\ 0 & 1 \end{pmatrix} \Rightarrow P_1$ is true.	B1		
		Assume P_k is true for some integer k.	B1		
	Proves inductive step.	$\mathbf{A}^{k+1} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 3(2^k-1) \\ 0 & 1 \end{pmatrix}$	M1		
		$= \begin{pmatrix} 2^{k+1} & 3.2(2^k-1)+3 \\ 0 & 1 \end{pmatrix}$			
		$= \begin{pmatrix} 2^{k+1} & 3(2^{k+1}-1) \\ 0 & 1 \end{pmatrix}$	A1		
	States conclusion.	Since P_1 is true and $P_k \Rightarrow P_{k+1}$, hence by PMI P_n is true $\forall$ positive integers n.	A1	5	**[5]**

Q6.

	Proves base case.	$H_n: \dfrac{d^n}{dx^n}(e^x \sin x) = 2^{\frac{n}{2}} e^x \sin\left(x + \dfrac{n\pi}{4}\right)$			
		$\dfrac{d}{dx}(e^x \sin x) = \sin x e^x + e^x \cos x$	M1		
		$= \sqrt{2}e^x\left(\dfrac{\sin x}{\sqrt{2}} + \dfrac{\cos x}{\sqrt{2}}\right) = 2^{\frac{1}{2}} e^x \sin\left(x + \dfrac{\pi}{4}\right)$			
		$\Rightarrow H_1$ is true.	A1		
	States inductive hypothesis.	Assume H_k is true :	B1		
	Proves inductive step.	$\dfrac{d^{k+1}}{dx^{k+1}}(e^x \sin x) = 2^{\frac{k}{2}}\left\{e^x \sin\left(x + \dfrac{k\pi}{4}\right) + e^x \cos\left(x + \dfrac{k\pi}{4}\right)\right\}$	M1		
		$= 2^{\frac{k+1}{2}} e^x \left\{\dfrac{1}{\sqrt{2}}\sin\left(x + \dfrac{k\pi}{4}\right) + \dfrac{1}{\sqrt{2}}\cos\left(x + \dfrac{k\pi}{4}\right)\right\}$	A1		
		$= 2^{\frac{k+1}{2}} e^x \left\{\sin\left(x + \dfrac{k\pi}{4} + \dfrac{\pi}{4}\right)\right\}$			
		$= 2^{\frac{k+1}{2}} e^x \sin\left(x + \dfrac{(k+1)\pi}{4}\right)$	A1		
	States conclusion.	$\therefore H_k \Rightarrow H_{k+1}$ Hence true for all positive integers by PMI	A1	7	[7]

Q7.

	States proposition.	$P_n : \frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right) = (-1)^n \frac{n!2^n}{(2x+3)^{n+1}}$			
	Proves base case.	$\frac{d}{dx}\left(\frac{1}{2x+3}\right) = (-1)(2x+3)^{-2} \times 2$	M1		
		$= (-1)\frac{1!\times 2}{(2x+3)^2} \Rightarrow P_1$ is true.	A1		
	States inductive hypothesis.	Assume P_k is true.			
		i.e. $\frac{d^k}{dx^k}\left(\frac{1}{2x+3}\right) = (-1)^k \frac{k!2^k}{(2x+3)^{k+1}}$	B1		
	Shows $P_k \Rightarrow P_{k+1}$.	$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{2x+3}\right) = (-1)^{k+1} \frac{2(k+1)k!2^k}{(2x+3)^{k+2}}$	M1		
		$= (-1)^{k+1} \frac{(k+1)!2^{k+1}}{(2x+3)^{k+2}}$	A1		
		$\therefore P_k \Rightarrow P_{k+1}$			
	States conclusion.	Since P_1 is true and $P_k \Rightarrow P_{k+1}$, hence by the principle of mathematical induction P_n is true $\forall n \in Z^+$.	A1	6	[6]

Q8.

	Proposition.	$H_N : \; S_N = 1 - \frac{1}{(N+1)!}$			
	Proves base case.	$S_1 = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2!} \Rightarrow H_1$ is true.	B1		
	States inductive hypothesis.	H_k : Assume $S_k = 1 - \frac{1}{(k+1)!}$ is true.	B1		
	Proves inductive step.	$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)! - (k+2) + (k+1)}{(k+2)!}$	M1		
		$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+2)!} \quad \therefore H_k \Rightarrow H_{k+1}$.	A1		
	States conclusion.	$\therefore$ (By PMI H_n is) true for all positive integers N.	A1	5	[5]

Q9.			
		$\phi(1) = 5 \times 5 - 1 = 24$ which is divisible by 8 $\Rightarrow H_1$ is true.	B1
		Assume P_k is true for some positive integer $k \Rightarrow \phi(k) = 8l$	B1
		$\phi(k+1) - \phi(k) = 5^{k+1}(4k+5) - 1 - 5^k(4k+1) + 1$	M1
		$= 5^k(20k + 25 - 4k - 1)$	A1
		$= 5^k(16k + 24) = 8m$	A1
		$\therefore \phi(k+1) = 8(l+m)$	A1
		Hence, by PMI, true for all positive integers n. (CWO – all previous marks required.)	A1
			[7]
		Alternatively	
		$\phi(k+1) = 5^{k+1}(4k+5) - 1$	
		$= 5.(4k.5^k) + 25.5^k - 1$	
		$= 5(8l - 5^k + 1) + 25.5^k - 1$	(M1A1)
		$= 40l + 20.5^k + 4$	
		$= 40l + 24.5^k - 4.5^k + 4$	(A1)
		$= 40l + 24.5^k - 4(5^k - 1)$	
		$= 40l + 24.5^k - 4(8l - 4k.5^k)$	
		$= 8l + 24.5^k + 16k.5^k$	(A1)
		$= 8m$	