Proof by induction 2

Q1. The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$ for every positive integer n.

Prove by mathematical induction that $a_n > 5$ for every positive integer n. [5]

Prove also that $a_n > a_{n+1}$ for every positive integer n. [2]

Q2. Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. [6]

Q3. It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that an n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals, where $n \geqslant 3$. [6]

Q4. Using factorials, show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. [2]

Hence prove by mathematical induction that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}x^n$$

for every positive integer n. [4]

Q5. **(i)** Show that $\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(x^{n+1}\ln x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n + (n+1)x^n \ln x)$. [2]

(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$ [5]

Q6. The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 3$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. [5]

(ii) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

Proof by induction 2

Q7. It is given that $y = e^x u$, where u is a function of x. The rth derivatives $\frac{d^r y}{dx^r}$ and $\frac{d^r u}{dx^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^x\left(\binom{n}{0}u + \binom{n}{1}u^{(1)} + \binom{n}{2}u^{(2)} + \ldots + \binom{n}{r}u^{(r)} + \ldots + \binom{n}{n}u^{(n)}\right).$$ [8]

[You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.]

Q8. **(a)** Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4 + r^2\right) = \tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]
