Proof by induction 2 - MS

Q1.

$a_1 > 5$ (given) $\Rightarrow$ H_1 is true.	B1
Assume H_k is true for some positive integer k, i.e. $a_k = 5 + \delta$, where $\delta > 0$.	B1
$a_{k+1} - 5 = \dfrac{4a_k^2 + 25}{5a_k} - 5 = \dfrac{4a_k^2 + 25 - 25a_k}{5a_k} = \dfrac{(4a_k - 5)(a_k - 5)}{5a_k} > 0$, $\Rightarrow a_{k+1} > 5$	M1A1
Or	
$a_{k+1} = \dfrac{4}{5}(5+\delta) + \dfrac{5}{5+\delta}$, $= 4 + \dfrac{4}{5}\delta + (1 - \dfrac{\delta}{5} + \dfrac{\delta^2}{25} - \ldots)$ for $0 < \delta < 5$	(M1) (A1)
$= 5 + \dfrac{3}{5}\delta + 0(\delta^2) \geqslant a_{k+1} > 5$, ($\delta \geqslant 5$ is trivial).	
$H_k \Rightarrow H_{k+1}$ and H_1 is true, hence by mathematical induction, the result is true for all $n \in \mathbf{Z}^+$ (N.B. The minimum requirement is ‘true for all positive integers’.)	A1 (5)
$a_{k+1} - a_k = \dfrac{5}{a_k} - \dfrac{1}{5}a_k$	M1
$\dfrac{5}{a_k} < 1$ and $\dfrac{1}{5}a_k > 1 \Rightarrow a_{k+1} - a_k < 0 \Rightarrow a_{k+1} < a_k$	A1 (2)
	Total: 7

Q2.

For $n = 1$ $10 + 192 + 5 = 207 = 9 \times 23 \Rightarrow H_1$ is true.	**B1**
Assume H_k is true for some positive integer $k \Rightarrow 10^n + 3.4^{n+2} + 5 = 9\alpha$	**B1**
Let $f(n) = 10^n + 3.4^{n+2} + 5$	
Hence $f(n+1) - f(n) = 10^n(10-1) + 3.4^{n+2}(4-1)$	**M1**
$= 9(10^n + 4^{n+2})$	
$= 9\beta$	**A1**
Hence $f(n+1)(= 9(\beta + \alpha)) \Rightarrow H_{k+1}$ is true	**A1**
H_1 is true and $H_k \Rightarrow H_{k+1}$, hence by PMI H_n is true for all positive integers n.	**A1**
N.B. Or can show $f(n+1) = 9(10\alpha - 2.4^{n+2} - 5)$ for **M1A1A1**. (3rd, 4th & 5th marks)	[6]

Q3.

With $n = 3$, $\dfrac{1}{2}n(n-3) = 0$	**M1**
A triangle has no diagonals $\Rightarrow H_3$ is true.	**A1**
Assume H_k is true: A k-gon has $\dfrac{1}{2}k(k-3)$ diagonals for some integer ≥ 3	**B1**
Adding an extra vertex, a further $(k-1)$ diagonals can be drawn.	**M1**
$\dfrac{1}{2}k(k-3) + k - 1 = \dfrac{k^2 - 3k + 2k - 2}{2} = \dfrac{(k+1)(k-2)}{2}$	**A1**
$= \dfrac{1}{2}(k+1)(k+1-3)$ (So $H_k \Rightarrow H_{k+1}$)	**A1**
$\Rightarrow H_n$ is true for all integers $n \geqslant 3$.	[6]

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Q4.

4	$\binom{n}{r-1}+\binom{n}{r}=\frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{r!(n-r)!}=\frac{n!}{(r-1)!(n-r)!}\left(\frac{1}{n-r+1}+\frac{1}{r}\right)$	M1	
	$=\frac{n!}{(r-1)!(n-r)!}\left(\frac{r+n-r+1}{r(n-r+1)}\right)=\frac{(n+1)!}{r!(n-r+1)!}=\binom{n+1}{r}$	A1	[2]
	$(a+x)^1=\binom{1}{0}a+\binom{1}{1}x=a+x\Rightarrow H_1$ is true.	B1	
	Assume H_k is true,i.e. $(a+x)^k=\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x+\ldots+\binom{k}{r}a^{k-r}x^r+\ldots+\binom{k}{k}x^k$	B1	
	Multiplying by $(a+x)$, the coefficient of $a^{k-r+1}x^r$ is: $\binom{k}{r-1}+\binom{k}{r}=\binom{k+1}{r}$	M1	
	$\Rightarrow H_{k+1}$ is true.		
	Hence H_n is true for all positive integers.	A1	[4]

Q5.

(i)	$\frac{d^{n+1}}{dx^{n+1}}\left(x^{n+1}\ln x\right)=\frac{d^n}{dx^n}\left(x^{n+1}.\frac{1}{x}+(n+1)x^n\ln x\right)=$ $\frac{d^n}{dx^n}\left(x^n+(n+1)x^n\ln x\right)$	M1A1	AG
		2	
(ii)	Assume H_k is true $\Rightarrow \frac{d^k}{dx^k}\left(x^k\ln x\right)=k!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k}\right\}$	B1	Statement of H_k seen
	$\frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\ln x\right)=\frac{d^k}{dx^k}\left(x^k+[k+1]x^k\ln x\right)$	M1	
	$=k!+[k+1]k!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k}\right\}$	A1	
	$=(k+1)!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k+1}\right\}\Rightarrow H_{k+1}$ is true	A1	
	Check H_1 is true **and** H_k is true $\Rightarrow H_{k+1}$ is true; hence, by PMI, H_n is true for all positive integers n.	A1	
		5	

Q6.

(iii)	$\sum_{n=1}^{N}\ln(u_n-3)=\sum_{n=1}^{N}\ln\left(4\left(\frac{5}{4}\right)^n\right)$ $=\left(\ln\frac{5}{4}\right)\sum_{n=1}^{N}n+\sum_{n=1}^{N}\ln 4$	M1	Alt method: $\sum_{n=1}^{N}\ln(u_n-3)=\ln\prod 4\left(\frac{5}{4}\right)^n$ M1 $=\ln 4^N\prod_1^N\left(\frac{5}{4}\right)^n$ $=N\ln 4+\ln\left(\frac{5}{4}\right)^{\Sigma n}$
	$=\frac{1}{2}N(N+1)\ln\frac{5}{4}+N\ln 4$ Use $\sum_{n=1}^{N}n=\frac{1}{2}N(N+1)$.	M1	$=N\ln 4+\frac{N(N+1)}{2}\ln(\frac{5}{4})$ M1
	$=N^2\ln\frac{\sqrt{5}}{2}+N\ln\left(2\sqrt{5}\right)\Rightarrow a=\frac{\sqrt{5}}{2},\ b=2\sqrt{5}$ oe Alt method: Writes series as an AP M1, uses summation formula M1 Correct answer A1	A1	$=N^2\ln\frac{\sqrt{5}}{2}+N\ln\left(2\sqrt{5}\right)\Rightarrow a=\frac{\sqrt{5}}{2},\ b=2\sqrt{5}$ A1

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Q7.

	$y^{(1)} = e^x u^{(1)} + ue^x = e^x\left(\binom{1}{0}u + \binom{1}{1}u^{(1)}\right) \Rightarrow H_1$ is true	M1A1	Shows base case using product rule
	Assume that $H_k : y^{(k)} = e^x\left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots + \binom{k}{k}u^{(k)}\right)$	B1	States inductive hypothesis.
	Then $y^{(k+1)} = e^x\left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots + \binom{k}{k}u^{(k)}\right) +$	M1	Differentiates using product rule
	$e^x\left(\binom{k}{0}u^{(1)} + \binom{k}{1}u^{(2)} + \cdots + \binom{k}{r}u^{(r+1)} + \cdots + \binom{k}{k}u^{(k+1)}\right)$ $= e^x\left(\binom{k}{0}u + \ldots\left(\binom{k}{r} + \binom{k}{r-1}\right)u^r + \ldots\binom{k}{k}u^{(k+1)}\right)$	M1A1	Shows application of $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.
	$= e^x\left(\binom{k+1}{0}u + \ldots\binom{k+1}{r}u^r + \ldots\binom{k+1}{k+1}u^{(k+1)}\right)$	B1	Shows reasoning for first and last term correctly
	So H_k *implies* H_{k+1} so, by induction, H_n is true for all $n \geqslant 1$.	A1	States conclusion.
		8	

Q8.

(a)	$5\times1^4 + 1^2 = \frac{1}{2}(2)^2(2+1)[=6]$ so H_1 is true.	B1	Checks base case.
	Assume that $\sum_{r=1}^{k}\left[\left(5r^4 + r^2\right)\right] = \frac{1}{2}k^2(k+1)^2(2k+1)$	B1	States inductive hypothesis [for some k] including the algebraic form. If says for **ALL** k, then B0.
	$\sum_{r=1}^{k+1}\left[\left(5r^4 + r^2\right)\right] = \frac{1}{2}k^2(k+1)^2(2k+1) + 5(k+1)^4 + (k+1)^2$	M1	Considers sum to $k+1$.
	$\frac{1}{2}(k+1)^2\left(2k^3 + k^2 + 10(k+1)^2 + 2\right)$	M1	Take out the factor of $(k+1)^2$ OR expands the summation expression and the target expression for $k+1$ and collects like terms for both.
	$\frac{1}{2}(k+1)^2(2k^3 + 11k^2 + 20k + 12) = \frac{1}{2}(k+1)^2(k+2)^2(2k+3)$	A1	Factorises or having expanded, checks explicitly. At least one intermediate step seen following the award of M1 before reaching the answer.
	So H_{k+1} is true. By induction, H_n is true for all positive integers *n*.	A1	States conclusion. Implication must be clearly expressed.
		6	
(b)	$5\sum_{r=1}^{n}r^4 + \frac{1}{6}n(n+1)(2n+1) = \frac{1}{2}n^2(n+1)^2(2n+1)$	M1	Uses correct formula for $\sum r^2$.
	$[5]\sum_{r=1}^{n}r^4 = \frac{1}{6}n(n+1)(2n+1)(3n(n+1)-1)$	M1	Makes $\sum r^4$ the subject and takes out all linear factors and the remaining term is of correct form.
	$\sum_{r=1}^{n}r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$	A1	CAO
		3	