Roots of Polynomial Equations

Revision Exercise 2 (Cubic Polynomial Equations)

1. The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $\alpha^3, \beta^3, \gamma^3$ is

$$y^3 - 3y^2 + 4y - 1 = 0. \quad [4]$$

Hence find the value of $\alpha^6 + \beta^6 + \gamma^6$. [3]

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2. The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ. Use the substitution $y = \dfrac{1}{1+x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\dfrac{1}{\alpha+1}, \dfrac{1}{\beta+1}$ and $\dfrac{1}{\gamma+1}$. [2]

For the cases $n = 1$ and $n = 2$, find the value of

$$\frac{1}{(\alpha+1)^n} + \frac{1}{(\beta+1)^n} + \frac{1}{(\gamma+1)^n}. \quad [2]$$

Deduce the value of $\dfrac{1}{(\alpha+1)^3} + \dfrac{1}{(\beta+1)^3} + \dfrac{1}{(\gamma+1)^3}$. [2]

Hence show that $\dfrac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \dfrac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \dfrac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \dfrac{73}{36}$. [3]

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3. The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are $\dfrac{\beta}{k}, \beta, k\beta$, where p, q, r, k and β are non-zero real constants. Show that $\beta = -\dfrac{q}{p}$. [4]

Deduce that $rp^3 = q^3$. [2]

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4. The equation

$$x^3 + 5x + 3 = 0$$

has roots α, β, γ. Use the substitution $x = -\dfrac{3}{y}$ to find a cubic equation in y and show that the roots of this equation are $\beta\gamma, \gamma\alpha, \alpha\beta$. [4]

Find the exact values of $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3$. [5]

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5. The equation

$$x^3 + 5x + 3 = 0$$

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Find the exact values of $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3$. [5]

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6. The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Use the relation $x = \sqrt{y}$ to show that the equation

$$y^3 + 2y^2 + y - 1 = 0$$

has roots $\alpha^2, \beta^2, \gamma^2$. [2]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) Write down the value of S_2 and show that $S_4 = 2$. [3]

(ii) Find the values of S_6 and S_8. [4]
