Roots of Polynomial Equations 2 - MS

Q1.

$y=\frac{1}{x+1}$ $\therefore x=\frac{1-y}{y}$	M1	use in given cubic equation	
Gives $6y^3-7y^2+3y-1=0$ **AG**	A1		[2]
$n=1$: given expression = sum of roots = 7/6	B1		
$n=2$: $\sum\frac{1}{(\alpha+1)^2}=\left(\sum\frac{1}{(\alpha+1)}\right)^2-2\sum\text{"}\alpha\beta\text{"}=\frac{13}{36}$	B1		[2]
From cubic in y,			
$6\sum\left(\frac{1}{\alpha+1}\right)^3-7.\frac{13}{36}+3\left(\frac{7}{6}\right)-3=0$	M1		
$\sum\left(\frac{1}{\alpha+1}\right)^3=73/216$	A1		[2]
LHS = $\sum\left(\frac{(\beta+1)(\gamma+1)(\alpha+1}{(\alpha+1)^3}\right)$	M1		
$=\left(\frac{1}{6}\right)^{-1}\times\frac{73}{216}$	M1	recognise product of roots	
= 73/36 **AG**	A1		[3]

Q2.

Sum of roots.	$\frac{\beta+\beta k+\beta k^2}{k}=-p$	B1		
Sum of products in pairs.	$\frac{\beta^2}{k}+k\beta^2+\beta^2=q$	B1		
Factorises.	$\Rightarrow\beta\left(\frac{k^2+k+1}{k}\right)=-p$ and $\beta^2\left(\frac{k^2+k+1}{k}\right)=q$	M1		
	$\Rightarrow\beta=-\frac{q}{p}$ (AG)	A1	4	
Product of roots.	$\beta^3=-r$	B1		
	$\Rightarrow-\frac{q^3}{p^3}=-r\Rightarrow rp^3=q^3$ (AG)	B1	2	**[6]**

Q3.

Uses $\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$	$36=38+2\sum\alpha\beta\Rightarrow\sum\alpha\beta=-1$	M1A1		
States equation with required roots.	$\therefore t^3+6t^2-t-30=0$ is the required equation.	A1	3	
Factorises	$\Rightarrow(t-2)(t+3)(t+5)=0$	M1A1		
Gives values of α, β, γ.	Hence α, β, and γ are 2, –3 and –5 (in any order). N.B. Answers written down with no working get B1.	A1	3	**[6]**

Roots of Polynomial Equations 2 - MS

Q4.

	(N.B. Not α, β, γ)	Let roots be α, α, and β.			
	Writes down sum of roots,	(1) $2\alpha + \beta = 0$	M1		
	sum of products in pairs	(2) $2\alpha\beta + \alpha^2 = p$	A1		
	and product of roots.	(3) $\alpha^2\beta = -q$	A1		
		From (1) $\beta = -2\alpha$			
	Eliminates β (or α).	(2) $\Rightarrow -4\alpha^2 + \alpha^2 = p \Rightarrow p = -3\alpha^2$			
		(3) $\Rightarrow -2\alpha^3 = -q \Rightarrow q = 2\alpha^3$	M1		
	Equates power of α (or β)	$\alpha^6 = \left(-\frac{p}{3}\right)^3 = \left(\frac{q}{2}\right)^2 \Rightarrow 4p^3 + 27q^2 = 0$ (AG)	A1	5	**[5]**

Q5.

	Uses	$\sum\alpha = -5 \qquad \sum\alpha\beta = -3$	B1		
	$\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$	$\sum\alpha^2 = (-5)^2 - 2\times(-3) = 31$	M1A1	3	
	Evaluates determinant.	Det $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix} = 1 - (\alpha^2 + \beta^2 + \gamma^2) + 2\alpha\beta\gamma$	M1A1		
		$\alpha\beta\gamma = -(-15) = 15$			
		$\Rightarrow 1 - 31 + 2\times 15$	M1		
	Shows it is zero.	$= 0 \Rightarrow$ matrix is singular.	A1	4	**[7]**

Q6.

	States $\sum\alpha$ and $\sum\alpha\beta$	$\sum\alpha = 7 \quad \sum\alpha\beta = 2$	B1		
	Uses formula for correctly.	$\sum\alpha^2 = 7^2 - 2\times 2 = 45$	B1	2	
	Uses formula for $\sum\alpha^3$ to obtain result.	$\sum\alpha^3 = 7\sum\alpha^2 - 2\sum\alpha + 9$ $= 315\text{-}14\text{+}9 = 310$	M1 A1A1	3	**[5]**

Roots of Polynomial Equations 2 - MS

Q7.

	EITHER Substitute α into equation. Multiply by α^n. Obtain result.	α is a root $\Rightarrow \alpha^4 - 3\alpha^2 + 5\alpha - 2 = 0$ $\Rightarrow \alpha^{n+4} - 3\alpha^{n+2} + 5\alpha^{n+1} - 2\alpha^n = 0$ Repeat for β, γ, δ and sum $\Rightarrow S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0$ (AG)	M1 A1	2	
(i)	Uses $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$ Finds S_4 from formula.	$S_2 = 0 - 2\times(-3) = 6$ $S_4 = 3\times 6 - 5\times 0 + 2\times 4 = 26$	B1 M1A1	3	
(ii)	$S_{-1} = \dfrac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}$ Finds S_3 from formula. Finds S_5 from formula.	$S_{-1} = \dfrac{-5}{-2} = \dfrac{5}{2}$ $S_3 = 3\times 0 - 5\times 4 + 2\times\dfrac{5}{2} = -15$ $S_5 = 3\times(-15) - 5\times 6 + 2\times 0 = -75$	M1A1 M1A1 M1A1	6	
		$\sum\alpha^2\beta^3 = S_2S_3 - S_5$ $= 6\times(-15) - (-75) = -15$	M1 M1A1	3	**[14]**

Q8.

		$(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = \alpha^2+\beta^2+\gamma^2 \Rightarrow \sum\alpha\beta = 1$	M1A1	2	
		Either Required equation is $x^3 - 4x^2 + x + c = 0$ $\Rightarrow \sum\alpha^3 - 4\sum\alpha^2 + 4 + 3c = 0$ $\Rightarrow 3c = 56 - 34 - 4 = 18 \Rightarrow c = 6$ (AG)	M1 M1 A1		
		Or $\alpha^3+\beta^3+\gamma^3 - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$ (or some other appropriate identity, e.g. $(\alpha+\beta+\gamma)^3 = \alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 3\alpha\beta\gamma$) $\Rightarrow \ldots \Rightarrow \alpha\beta\gamma = -6$ $\Rightarrow x^3 - 4x^2 + x + 6 = 0$ (AG) $\Rightarrow (x+1)(x-2)(x-3) = 0 \Rightarrow x = -1, 2, 3.$	(M1) (M1A1) A1 M1A1	6	**[8]**