Vectors 1 MS

Q1.

7 **(i)** Solves any 2 of the equations:

$4+2\lambda = 4+\mu$, $-2+\lambda = -5-\mu$, $-4\lambda = 2-\mu$

to obtain $\lambda = -1$, $\mu = -2$ M1A1

Checks consistency with the third equation A1

[3]

(ii) $P = |(\mathbf{i}-2\mathbf{j}+2\mathbf{k}).(5\mathbf{i}+2\mathbf{j}+3\mathbf{k})/\sqrt{38}|$ M1A1

$=7/\sqrt{38} = 1.14$ A1

[3]

OR

$\mathbf{n} = -5\mathbf{i}-2\mathbf{j}-3\mathbf{k}$ M1

Plane is $5x+2y+3z=16$ A1

$$P = \frac{15-10+18-16}{\sqrt{5^2+2^2+3^2}} = \frac{7}{\sqrt{38}}$$ A1

OR

Plane is $5x+2y+3z=16$ (as above) M1A1

Sub. general pt on perpendicular $\begin{pmatrix}3+5t\\-5+2t\\6+3t\end{pmatrix} \Rightarrow t = -\frac{7}{38}$

$$\Rightarrow P = \left|\begin{pmatrix}5t\\2t\\3t\end{pmatrix}\right| = 1.14$$ A1

(iii) $(\mathbf{i}-2\mathbf{j}+2\mathbf{k})\times(2\mathbf{i}+\mathbf{j}-4\mathbf{k}) = 6\mathbf{i}+8\mathbf{j}+5\mathbf{k}$

OR $(\mathbf{i}+3\mathbf{j}-6\mathbf{k})\times(2\mathbf{i}+\mathbf{j}-4\mathbf{k}) = -6\mathbf{i}-8\mathbf{j}-5\mathbf{k}$, etc. B1

$d = |6\mathbf{i}+8\mathbf{j}+5\mathbf{k}|/\sqrt{21} = \sqrt{125/21} = 2.44$ M1A1A1

[4]

OR

Let Q be the foot of the perpendicular from P to l, and A be the known point on l_1

$$AQ = \left|(\mathbf{i}-2\mathbf{j}+2\mathbf{k}).\frac{(2\mathbf{i}+\mathbf{j}-4\mathbf{k})}{\sqrt{21}}\right| = \frac{8}{\sqrt{21}}$$ M1A1

$AP^2 = 1^2+(-2)^2+2^2 = 9$ B1

$$PQ^2 = 9 - \frac{64}{21} = \frac{125}{21} \Rightarrow PQ = \frac{5\sqrt{5}}{21}$$ A1

OR

$$\overrightarrow{PQ} = \begin{pmatrix}4+2t\\-2+t\\-4t\end{pmatrix} - \begin{pmatrix}3\\-5\\6\end{pmatrix} = \begin{pmatrix}1+2t\\3+t\\-6-4t\end{pmatrix} \therefore \begin{pmatrix}1+2t\\3+t\\-6-4t\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-4\end{pmatrix} = 0$$ M1

$$\Rightarrow t = -\frac{29}{21}$$ A1

$$\overrightarrow{PQ} = \frac{1}{21}\begin{pmatrix}-37\\34\\-10\end{pmatrix} \Rightarrow \left|\overrightarrow{PQ}\right| = \frac{1}{21}\sqrt{37^2+34^2+10^2} = 2.44$$ M1A1

Vectors 1 MS

Q2.

12 EITHER

(i) $(\mathbf{i}+\mathbf{j})\times(\mathbf{j}+2\mathbf{k}) = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ M1A1

$PQ = (4\mathbf{i}+\mathbf{j}+3\mathbf{k}).(2\mathbf{i}-2\mathbf{j}+\mathbf{k})/3 = 9/3 = 3$ M1A1

[4]

(ii) $(4+\lambda)/2 = (1+\lambda-\mu)/(-2) = (3-2\mu)/1$ (AEF)

OR $(4+\lambda)+(1+\lambda-\mu)=0$, $(1+\lambda-\mu+2(3-2\mu)=0$, both M1A1

$\Rightarrow \ldots \Rightarrow \mu = 1$ M1A1

Position vector of Q is $-\mathbf{i}+\mathbf{j}+\mathbf{k}$ A1

[5]

Parts **(i)** and **(ii)** together:

$$\overrightarrow{PQ} = \begin{pmatrix} 4+\lambda \\ 1+\lambda-\mu \\ 3-2\mu \end{pmatrix}$$ B1

$$\overrightarrow{PQ}.\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow 5+2\lambda-\mu=0$$ M1A1

$$\overrightarrow{PQ}.\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0 \Rightarrow 7+\lambda-5\mu=0$$ A1

$\lambda = -2,\ \mu = 1$ M1A1

$$\overrightarrow{PQ}.\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \left|\overrightarrow{PQ}\right| = 3$$ M1A1

$$\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$ A1

[9]

Vectors 1 MS

(iii) $(\mathbf{i}+\mathbf{j}) \times (4\mathbf{i}+\mathbf{j}+3\mathbf{k}) = 3\mathbf{i}-3\mathbf{j}-3\mathbf{k}$ — M1A1

$p_2 = \left|[(\mathbf{i}-\mathbf{j}-\mathbf{k}).(4\mathbf{i}+\mathbf{k})]/\sqrt{3}\right|$ — M1

$= \ldots = \sqrt{3}$ — A1

[4]

for final 2 marks
$\pi : x - y - z = 0$

perpendicular distance $= \left|\left(\dfrac{-1-1-1}{\sqrt{1^2+1^2+1^2}}\right)\right| = \sqrt{3}$

or any other method.
M1 is for a complete strategy and A1 for $\sqrt{3}$.

Q3.

6 (i)	Uses vector product to find vector perpendicular to both lines.	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 0 \\ 4 & 0 & -1 \end{vmatrix} = -3\mathbf{i}+4\mathbf{j}-12\mathbf{j}$	M1A1		
	Finds ***BA*** and its scalar product with unit perpendicular vector.	$\boldsymbol{BA} = \mathbf{i}+10\mathbf{j}-11\mathbf{j}$ perp.dist. $= \dfrac{-3\times1+4\times10+12\times11}{\sqrt{3^2+4^2+12^2}} = \dfrac{169}{\sqrt{169}} = 13$ $= 13$ (AG) (No penalty for sign errors made in $\mathbf{n}$, which lead to correct result.)	M1A1	4	
(ii)		$\mathbf{p} = \begin{pmatrix} 8+4\lambda \\ 8+3\lambda \\ -7 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 7+4\mu \\ -2 \\ 4-\mu \end{pmatrix}$			
		$\mathbf{PQ} = \begin{pmatrix} -1-4\lambda+4\mu \\ -10-3\lambda \\ 11-\mu \end{pmatrix} = t\begin{pmatrix} -3 \\ 4 \\ -12 \end{pmatrix}$	B1 M1A1		
		$\Rightarrow t=-1 \quad \lambda=-2 \quad \mu=-1$	M1		
		$\mathbf{p} = 2\mathbf{j}-7\mathbf{k} \qquad \mathbf{q} = 3\mathbf{i}-2\mathbf{j}+5\mathbf{k}$ (Award B1B1B1 if t assumed to be ±1 i.e. 3/5)	A1	5	**[9]**

Vectors 1 MS

6(ii)	**Alternative Solution:**				
	Finds two parameter representation for **PQ**	$\mathbf{p} = \begin{pmatrix} 8+4\lambda \\ 8+3\lambda \\ -7 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 7+4\mu \\ -2 \\ 4-\mu \end{pmatrix}$			
	Uses scalar product between **PQ** and direction vector of at least one line and equates to zero	$\mathbf{PQ} = \begin{pmatrix} -1-4\lambda+4\mu \\ -10-3\lambda \\ 11-\mu \end{pmatrix}$	B1		
		$16\mu - 25\lambda = 34$	M1A1		
		$17\mu - 16\lambda = 15$	A1		
	Solves simultaneously.	$\mu = -1 \quad \lambda = -2$	M1A1		
	Obtains position vectors for *P* and *Q*	$\mathbf{p} = 2\mathbf{j} - 7\mathbf{k} \quad \mathbf{q} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$	A1	7	
(i)	Obtains length of *PQ*.	$\sqrt{3^2 + (-4)^2 + 12^2} = 13$ (AG)	M1A1	2	[9]

Q4.

10	Uses vector product to find normal to plane.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & 6 & 1 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$	M1A1		
	Uses **r.n** = constant. Obtains cartesian equation of plane.	Equation of plane: $5x - 3y - 2z = \text{constant}$ $30 - 15 - 8 = 7$ $5x - 3y - 2z = 7$	M1 A1	4	
		Alternatively:			
		$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$	(M1)		
		$x = 6 + \lambda + 4\mu$ $y = 5 + \lambda + 6\mu$ $z = 4 + \lambda + \mu$	(A1)		
		Eliminates λ and μ.	(M1)		
		Obtains $5x - 3y - 2z = 7$	(A1)		

Vectors 1 MS

10 Contd.	Finds equation of perpendicular to plane through given point.	Equation of perpendicular: $\mathbf{r} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k} + t(5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$	M1		
	Finds value of parameter at point in plane. Obtains foot of perpendicular.	$5(1+5t) - 3(10-3t) - 2(3-2t) = 7$ $\Rightarrow t = 1$ Foot of perpendicular is $6\mathbf{i} + 7\mathbf{j} + \mathbf{k}$.	M1 A1 A1	4	
	Alternatively: Form sufficient equations, using orthogonality. Two will suffice if foot of perpendicular is expressed using parametric equation of plane.	Let foot of perpendicular be $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and using othogonality: $\begin{pmatrix} a-1 \\ b-10 \\ c-3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow a+b+c = 14$ $\begin{pmatrix} a-1 \\ b-10 \\ c-3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix} = 0 \Rightarrow 4a+6b+c = 67$ $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ lies in plane of l_1 and l_2: $5a - 3b - 2c = 7$ $\Rightarrow 6\mathbf{i} + 7\mathbf{j} + \mathbf{k}$	 (M1A1) (M1A1)		
	Finds direction of common perpendicular.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$	M1A1		
	Forms vector between known points on l_1 and l_3.	$\begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$			
	Finds shortest distance by projection.	$\frac{1}{\sqrt{16+1+25}} \left| \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \right| = \frac{10}{\sqrt{42}}$ $(=1.54)$	M1A1 A1	5	[13]

Vectors 1 MS

10 Contd.	**Alternative for last part:**	Let P be on l_1 and Q be on l_3. $\mathbf{p} = \begin{pmatrix} 6+\lambda \\ 5+\lambda \\ 4+\lambda \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1+2v \\ 10-3v \\ 3+v \end{pmatrix}$ $\Rightarrow \overrightarrow{PQ} = \begin{pmatrix} -5-\lambda+2v \\ 5-\lambda-3v \\ -1-\lambda-3v \end{pmatrix}$	(M1)		
		Uses orthogonality conditions: $\Rightarrow \overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow -1-3\lambda = 0 \Rightarrow \lambda = -\frac{1}{3}$	(M1)		
		$\overrightarrow{PQ} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \Rightarrow -26+14v = 0 \Rightarrow v = \frac{13}{7}$	(A1)		
		$\Rightarrow \overrightarrow{PQ} = \frac{1}{21}\begin{pmatrix} -20 \\ -5 \\ 25 \end{pmatrix}$	(A1)		
		$\Rightarrow \left\|\overrightarrow{PQ}\right\| = \frac{5}{21}\sqrt{4^2+1^2+5^2} = \frac{5}{21}\sqrt{42}$	(A1)	(5)	

Vectors 1 MS

Q5.

9	Finds normal to plane.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -3 & 4 \end{vmatrix} = \mathbf{i} - 9\mathbf{j} - 7\mathbf{k}$	M1A1		
(i)	Deduces equation.	Π:: $x - 9y - 7z = \text{constant}$			
		Sub e.g. $(1, -1, 2) \Rightarrow$ constant = – 4			
		Π:: $x - 9y - 7z = -4$	M1A1	4	
	General point on line inserted in plane equation to find λ.	l: $x = 6 + 2\lambda \quad y = -2 + \lambda \quad z = 1 - 4\lambda$			
		Sub in $\Pi \Rightarrow 6 + 2\lambda + 18 - 9\lambda - 7 + 28\lambda = -4$	M1		
		$\Rightarrow \lambda = -1$	A1		
		Position vector of intersection is 4**i** – 3**j** + 5**k.**	A1	3	
(ii)	Distance of point from plane formula or triple scalar product method.	Either $\left\lvert \frac{6 + 18 - 7 + 4}{\sqrt{1 + 81 + 49}} \right\rvert$ Or $\frac{(2i + j - 4k).(i - 9j - 7k)}{\sqrt{1 + 81 + 49}}$	M1A1		
		$= \frac{21}{\sqrt{131}}$ (=1.83)	A1	3	
(iii)	Scalar product to find complement of angle.	$(2i + j - 4k).(i - 9j - 7k) = 21$	M1		
		$= \sqrt{4 + 1 + 16}\sqrt{1 + 81 + 49}\sin\theta$	A1		
		$\Rightarrow \sin\theta = \sqrt{\frac{21}{131}} \Rightarrow \theta = 23.6^0$ or 0.412 rad.	A1	3	**[13]**

Vectors 1 MS

Q6.

11	**OR** Obtains direction of common perpendicular.	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 0 & 1 & m-1 \end{vmatrix} = -m\mathbf{i} + 4(1-m)\mathbf{j} + 4\mathbf{k}$	M1A1		
	Uses result for shortest distance between lines.	$\dfrac{\left\lvert \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \begin{pmatrix} -m \\ 4-4m \\ 4 \end{pmatrix} \right\rvert}{\sqrt{m^2 + 16(1-2m+m^2)+16}} = 3$	M1A1		
	Solves equation.	$\Rightarrow \ldots \Rightarrow 19m^2 - 40m + 4 = 0$	A1		
		$\Rightarrow (19m-2)(m-2) = 0$	M1		
		$\Rightarrow m = 2$, since m is an integer. (AG)	A1	7	
	Finds relevant vectors.	$\mathbf{CA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{CD} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ or $\mathbf{AD} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$	B1		
	Use of cross-product.	$\dfrac{1}{\sqrt{17}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & -4 \end{vmatrix} = \dfrac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$	M1		
	Obtains shortest distance.	$\dfrac{1}{\sqrt{17}} \sqrt{4^2 + 1^2 + 1^2} = \sqrt{\dfrac{18}{17}}$ $(= 1.03)$	A1	3	
	Finds 2nd vector in BCD (**CD** may already have been found.)	$\mathbf{BC} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$	B1		
	Finds normal vector to BCD. (Normal to ACD already found.)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 0 & 1 & 1 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \sim 2\mathbf{i} + \mathbf{j} - \mathbf{k}$	M1		
	Finds angle between planes = angle between normal vectors.	$\cos\theta = \dfrac{(4\mathrm{i} - \mathrm{j} + \mathrm{k}) \cdot (2\mathrm{i} + \mathrm{j} - \mathrm{k})}{\sqrt{16+1+1}\sqrt{4+1+1}} = \dfrac{6}{\sqrt{18}\sqrt{6}} = \dfrac{1}{\sqrt{3}}$	M1		
		$\therefore$ Angle between planes $= \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (AG)	A1	4	**[14]**

Vectors 1 MS

11	**OR** **Alternatives for middle part:**				
	Or (a) Vector from D to any point on AC	$\begin{pmatrix} 1+t \\ -1 \\ -5-4t \end{pmatrix}$	(B1)		
	Uses orthogonality to obtain t.	$\begin{pmatrix} 1+t \\ -1 \\ -5-4t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = 0 \Rightarrow t = -\frac{21}{17}$	(M1)		
	Finds magnitude of perpendicular.	$\frac{1}{\sqrt{17}}\sqrt{4^2+1^2+1^2} = \sqrt{\frac{18}{17}}$ (= 1.03)	(A1)	(3)	
	Or (b) Finds length of AD (or CD)	$\lvert\overrightarrow{AD}\rvert = \sqrt{27}$	(B1)		
	Finds projection of AD (or CD) onto AC.	$\left\lvert \frac{\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}}{\sqrt{4^2+1^2}} \right\rvert = \frac{21}{\sqrt{17}}$	(M1)		
	Finds perpendicular by Pythagoras.	$\sqrt{27 - \frac{441}{17}} = \sqrt{\frac{18}{17}}$ (= 1.03)	(A1)	(3)	

Vectors 1 MS

Q7.

9	Finds vector normal to Π.	$\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 2\\ 3 & 1 & -2\end{vmatrix}=2\mathbf{i}+8\mathbf{j}+7\mathbf{k}$	M1A1		
	Dot product of this with general point on l_1.	$\begin{pmatrix}3+8t\\6+5t\\12-8t\end{pmatrix}\cdot\begin{pmatrix}2\\8\\7\end{pmatrix}=138$ or $\begin{pmatrix}8\\5\\-8\end{pmatrix}\cdot\begin{pmatrix}2\\8\\7\end{pmatrix}=0$	A1√		
	Deduces result.	Independent of $t \Rightarrow$ parallel, or $\Rightarrow$ parallel.	A1	4	
	Cartesian equation of Π.	Π: $2x+8y+7z=21$	B1		
	Substitutes general point of l_2.	Sub. $x=5+2s$, $y=-4-s$, $z=7+s$	M1		
	Finds value of parameter.	$\Rightarrow s=-2$	A1		
	Finds p.v. of intersection.	and line meets Π at point with p.v. $\mathbf{i}-2\mathbf{j}+5\mathbf{k}$	A1	4	
		Take (9,11,2) as A, (3,6,12) as B and let C be foot of perpendicular from A to l.			
		$AB=\sqrt{6^2+5^2+10^2}=\sqrt{161}$	B1		
	Finds distance from point to known point on l.	$BC=\dfrac{1}{\sqrt{(6^2\,5^2+8^2)}(6i+5j-10k)(8i+5j-8k)}=\dfrac{153}{\sqrt{153}}=\sqrt{153}$	M1		
			A1	4	[12]
	Finds distance along l from known point to foot of perpendicular from given point to l.	$AC=\sqrt{161-153}=\sqrt{8 \text{ or } 2\sqrt{2}}$ (= 2.83)	A1√		
	F.t. on non-hypotenuse side (must be real).	**Alternatively:**			
	Writes a set of three equations in three unknowns for the intersection of l with Π.	$5+2s=2+\lambda+3\mu$ $-4-s=3-2\lambda+\mu$ $7+s=-1+2\lambda-2\mu$	(B1)		
	Solves the set of equations.	$\Rightarrow s=-2$, $\Rightarrow \lambda=2$, $\mu=-1$	(M1A1)		
	Finds p.v. of intersection.	and line meets Π at point with p.v. $\mathbf{i}-2\mathbf{j}+5\mathbf{k}$	(A1)		