Complex Numbers 1 MS

Q1.

$(c+is)^5 = c^5 + 5c^4(is) + 10c^3(-s^2) + 10c^2(is)^3 + 5c(is)^4 + (is)^5$ M1

$\theta = 5c^4s - 10c^2s^3 + s^5$ M1

$= 5s(1-s^2)^2 - 10s^3(1-s^2) + s^5$ M1

$= 16s^5 - 20s^3 + 5s$ (AG) A1 OEW [4]

$x = \sin\theta \Rightarrow \sin 5\theta = -1/2$ M1

Roots are $\sin q\pi$ where $q = 7/30, 11/30, 31/30, 35/30, 43/30$ A3

A1 for 1 root: + A1 for 2 further roots: + A1 for completion [4]
CWO CWO

Alternative answers

$q = \frac{11}{30}, \frac{23}{30}, \frac{35}{30}, \frac{47}{30}, \frac{59}{30}$

or

$q = \frac{7}{30}, \frac{19}{30}, \frac{31}{30}, \frac{43}{30}, \frac{55}{30}$

Q2.

$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$	M1A1	use of de Moivre for $(c+is)^5$	
$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$	A1		
$\tan 5\theta = \frac{t^5 - 10t^3 + 5t}{1 - 10t^2 + 5t^4}$ **AG**	M1A1	intermediate step needed	[5]
$\tan 5\theta = 0 \Rightarrow \theta = \frac{n\pi}{5}$	M1		
Solutions $\tan\frac{n\pi}{5}$ for $n = 1, 2, 3, 4$	A1	justify values of n	[2]
Roots $\pm\tan\frac{\pi}{5}, \pm\tan\frac{2\pi}{5}$	B1		
Product of these roots = 5	M1		
$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$	A1		[3]

Complex Numbers 1 MS

Q3.

5	Binomial expansion and groups.	$(z+z^{-1})^4=(z^4+z^{-4})+4(z^2+z^{-2})+6$, where $z=(\cos\theta+i\sin\theta)$.	M1A1		
	Uses de M's Thm.	$(2\cos\theta)^4=2\cos 4\theta+8\cos 2\theta+6$	M1		
	Simplifies	$\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$	A1	4	
	Integrates result	$\int_0^{\frac{\pi}{4}}\cos^4 d\theta=\int_0^{\frac{\pi}{4}}\left(\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}\right)d\theta$	M1		
	correctly.	$=\left[\frac{\sin 4\theta}{32}+\frac{\sin 2\theta}{4}+\frac{3\theta}{8}\right]_0^{\frac{\pi}{4}}$	A1		
	Evaluates.	$=\frac{1}{4}+\frac{3\pi}{32}$	A1	3	**[7]**

Q4.

7	Expands and groups. Use of $z-z^{-1}$ and $z+z^{-1}$. Correctly.	$\left(z-z^{-1}\right)^6=\left(z^6+z^{-6}\right)-6\left(z^4+z^{-4}\right)+15\left(z^2+z^{-2}\right)-20$	M1A1 M1		
	Obtains result.	$(2i\sin\theta)^6=2\cos 6\theta-12\cos 4\theta+30\cos 2\theta-20$ $\sin^6\theta=\frac{1}{32}(10-15\cos 2\theta+6\cos 4\theta-\cos 6\theta)$ (Allow $p=10$, $q=-15$, $r=6$, $s=-1$)	A1A1 A1	6	
	Integrates correctly.	$\left[\frac{5\theta}{16}-\frac{15\sin 2\theta}{64}+\frac{3\sin 4\theta}{64}-\frac{\sin 6\theta}{192}\right]_0^{\frac{\pi}{4}}$	M1A1		
	Inserts limits and evaluates.	$\frac{5\pi}{64}-\frac{15}{64}+\frac{1}{192}=\frac{5\pi}{64}-\frac{11}{48}$ (SC: 1f power of 2 consistently wrong ¾ for 2nd part.)	M1A1	4	**[10]**

Complex Numbers 1 MS

Q5.

5	$\dfrac{z(z^n-1)}{z-1}$ (OE)	B1 [1]
	$\cos\theta+\cos 2\theta+\cos 3\theta+\ldots+\cos n\theta \quad = \text{Re}\left\{\dfrac{z\left(z^n-1\right)}{(z-1)}\right\}$	M1
	$= \text{Re}\left\{\dfrac{z^{\frac{1}{2}}\left(z^n-1\right)}{\left(z^{\frac{1}{2}}-z^{-\frac{1}{2}}\right)}\right\}$	M1
	$= \text{Re}\left\{\dfrac{z^{n+\frac{1}{2}}-z^{\frac{1}{2}}}{2\text{i}\sin\frac{1}{2}\theta}\right\}$	A1
	$= \dfrac{\sin\left(n+\frac{1}{2}\right)\theta-\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$	A1
	$= \dfrac{2\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{2\sin\frac{1}{2}\theta}$ (Both previous M marks req.)	M1A1
	$= \dfrac{\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$ (AG)	A1 [7]

Q6.

8(i)	$1=\dfrac{z^1-1}{z-1}$ So true when $n=1$.	**B1**	Shows base case.
	Assume that $1+z+\ldots+z^{k-1}=\dfrac{z^k-1}{z-1}$	**B1**	States inductive hypothesis.
	Then $1+z+\ldots+z^{k-1}+z^k=\dfrac{z^k-1}{z-1}+z^k=\dfrac{z^k-1+z^k(z-1)}{z-1}=\dfrac{z^{k+1}-1}{z-1}$, so true when $n=k+1$	**M1 A1**	Combines fractions.
	$H_k \to H_{k+1}$ Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		5	

Complex Numbers 1 MS

8(ii)	Since $\lvert z\rvert<1$, $\sum_{m=0}^{\infty} z^m = \frac{-1}{z-1}$	B1	States $\lvert z\rvert<1$ and uses formula for sum to infinity of geometric progression.
	$\sum_{m=1}^{\infty} 2^{-m}\sin m\theta = \mathrm{Im}\left(\sum_{m=0}^{\infty} z^m\right) = \mathrm{Im}\left(\frac{-1}{\frac{1}{2}\cos\theta + \mathrm{i}\frac{1}{2}\sin\theta - 1}\right)$	M1 A1	Uses de Moivre's theorem.
	$\mathrm{Im}\left(\frac{-\left(\frac{1}{2}\cos\theta - 1 - \mathrm{i}\frac{1}{2}\sin\theta\right)}{\frac{1}{4}\cos^2\theta - \cos\theta + 1 + \frac{1}{4}\sin^2\theta}\right)$	M1	Multiply numerator and denominator by conjugate.
	$\frac{\frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta} = \frac{2\sin\theta}{5 - 4\cos\theta}$	A1	States imaginary part, AG.
		5	

Q7.

3(i)	$\exp\left(\mathrm{i}\frac{2\pi k}{5}\right)$, $k = 0, \pm1, \pm2$	B2	B1 for 1 correct fifth root. B1 for all 5 distinct, correct roots (AEF). SCB1 if only arguments are given.
3(ii)	$z^5 = -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$	M1	Correctly solves quadratic in z^5
	$z^5 = \exp\left(\mathrm{i}2\pi\left(\frac{1}{3}+k\right)\right)$ or $\exp\left(\mathrm{i}2\pi\left(-\frac{1}{3}+k\right)\right)$	M1 A1	Writes in polar or exponential form and adds multiples of 2π
	$\exp\left(\pm\mathrm{i}\frac{2\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{4\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{8\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{2\pi}{3}\right)$, $\exp\left(\pm\mathrm{i}\frac{14\pi}{15}\right)$	A2	A1 for 5 distinct, correct roots. A1 for exactly 10 distinct, correct roots. Allow alternative exact values of θ such as $\theta = \frac{16\pi}{15}, \frac{4\pi}{3}, \frac{22\pi}{15}, \frac{26\pi}{15}, \frac{28\pi}{15}$.
	Alternative method for 3(ii)		
	$z^5 + z^{-5} = -1$	M1	Divides through by z^5
	$2\cos 5\theta = -1$	M1 A1	Applies de Moivre's theorem
	$\exp\left(\pm\mathrm{i}\frac{2\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{4\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{8\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{2\pi}{3}\right)$, $\exp\left(\pm\mathrm{i}\frac{14\pi}{15}\right)$	A2	A1 for 5 distinct, correct roots. A1 for exactly 10 distinct, correct roots. Allow alternative exact values of θ such as $\theta = \frac{16\pi}{15}, \frac{4\pi}{3}, \frac{22\pi}{15}, \frac{26\pi}{15}, \frac{28\pi}{15}$.
		5	

Complex Numbers 1 MS

Q8.

9(i)	Write $c = \cos\theta$, $s = \sin\theta$. $\cos 6\theta + i\sin 6\theta = (c + is)^6$	**M1**	Uses binomial theorem.
	$\Rightarrow \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$	**A1**	
	$c^6 - 15c^4s^2 + 15c^2s^4 - s^6 = c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$	**M1**	Uses $c^2 = 1 - s^2$.
	$= c^6 - 15c^4(1-c^2) + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6)$	**A1**	
	$= 32c^6 - 48c^4 + 18c^2 - 1$	**M1**	Divides numerator and denominator by c^6.
	$\Rightarrow \sec 6\theta = \dfrac{1}{32c^6 - 48c^4 + 18c^2 - 1} = \dfrac{\sec^6\theta}{32 - 48\sec^2\theta + 18\sec^4\theta - \sec^6\theta}$	**A1**	AG
		6	

9(ii)	$x^6 = 2(32 - 48x^2 + 18x^4 - x^6) \Rightarrow \dfrac{x^6}{32 - 48x^2 + 18x^4 - x^6} = 2$	**M1 A1**	Relates with equation in part (i).
	$\sec 6\theta = 2 \Rightarrow \cos 6\theta = \dfrac{1}{2}$	**M1**	Solves $\cos 6\theta = \dfrac{1}{2}$.
	$x = \sec\dfrac{\pi}{18}$	**A1**	Gives one correct solution.
	$x = \sec q\pi$, $q = \dfrac{5}{18}, \dfrac{7}{18}, \dfrac{11}{18}, \dfrac{13}{18}, \dfrac{17}{18}$	**A1**	Gives five other solutions. Allow different values of q as long as all six solutions are found.
		5	

Q9.

6(a)	$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$	1	**B1**	
	$= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$	2	**M1A1**	Binomial expansion
	$\tan 5\theta = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$	1	**M1**	
	$\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$	1	**A1**	AG Division of each term by c^5 clearly stated
		5		
6(b)	Roots of $\tan 5\theta = 0$ are $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$	1	**B1**	Allow inclusion of 0 here
	$t^4 - 10t^2 + 5 = 0$ has roots $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$	1	**B1**	
	$\left(t - \tan\frac{\pi}{5}\right)\left(t - \tan\frac{2\pi}{5}\right)\left(t - \tan\frac{3\pi}{5}\right)\left(t - \tan\frac{4\pi}{5}\right) = 0$	1	**M1**	
	Since $\tan\frac{4\pi}{5} = -\tan\frac{\pi}{5}$ and $\tan\frac{3\pi}{5} = -\tan\frac{2\pi}{5}$, $\left(t^2 - \tan^2\left(\frac{\pi}{5}\right)\right)\left(t^2 - \tan^2\left(\frac{2\pi}{5}\right)\right) = 0$	1	**M1**	
	So roots of $x^2 - 10x + 5 = 0$ are $\tan^2\left(\frac{\pi}{5}\right)$ and $\tan^2\left(\frac{2\pi}{5}\right)$	1	**A1**	AG
		5		

Complex Numbers 1 MS

Q10.

3(a)	$z^3 = -1 - \mathrm{i} = 2^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\frac{5}{4}\pi}$	**B1**
	$z_1 = 2^{\frac{1}{6}} \mathrm{e}^{\mathrm{i}\frac{5}{12}\pi}$	**M1 A1**
	$z_2 = 2^{\frac{1}{6}} \mathrm{e}^{\mathrm{i}\frac{13}{12}\pi}$, $z_3 = 2^{\frac{1}{6}} \mathrm{e}^{\mathrm{i}\frac{21}{12}\pi}$	**A1 A1**
		5
3(b)	$z_1^{3k} + z_2^{3k} + z_3^{3k} = 3\left(2^{\frac{1}{2}k} \mathrm{e}^{\mathrm{i}\frac{5}{4}k\pi}\right)$	**M1**
	$R = \lvert w \rvert = 3\left(2^{\frac{1}{2}k}\right)$	**A1**
	$\alpha = \frac{5}{4}k\pi$	**A1**
		3