Complex Numbers 2 MS

Q1.

8(a)	$z - z^{-1} = 2\mathrm{i}\sin\theta$	B1
	$(z-z^{-1})^6 = (z^6+z^{-6}) - 6(z^4+z^{-4}) + 15(z^2+z^{-2}) - 20$	M1 A1
	$(2\mathrm{i}\sin\theta)^6 = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$	M1 A1
	$\sin^6\theta = \frac{1}{32}(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10)$	A1
		6
8(b)	$\int_0^{\frac{1}{3}\pi} \cos^6 \frac{1}{4}x + \sin^6 \frac{1}{4}x \,\mathrm{d}x = \frac{1}{8}\int_0^{\frac{1}{3}\pi} 3\cos x + 5\,\mathrm{d}x$	M1 A1
	$\frac{1}{8}[3\sin x + 5x]_0^{\frac{1}{3}\pi} = \frac{1}{8}\left(\frac{5}{3}\pi + \frac{3}{2}\sqrt{3}\right)$.	M1 A1
		4
8(c)	$c = \cos\theta \Rightarrow 1 - c^2 = \sin^2\theta$	B1
	$16c^6 + 16(1-c^2)^3 - 13 = 0 \Rightarrow 6\cos 4\theta - 3 = 0$	M1 A1
	$4\theta = \frac{1}{3}\pi, \frac{5}{3}\pi, \frac{7}{3}\pi, \frac{11}{3}\pi$	M1
	$c = \cos\left(\frac{1}{12}\pi\right), \cos\left(\frac{5}{12}\pi\right), \cos\left(\frac{7}{12}\pi\right), \cos\left(\frac{11}{12}\pi\right)$	A1
		5

Q2.

7(a)	$z^2 + z^2(z^2) + \ldots + z^2(z^2)^{n-1} = \dfrac{z^2(z^{2n}-1)}{z^2-1}$	M1	Uses sum of geometric series.
	$\dfrac{z^2(z^{2n}-1)}{z^2-1} \times \dfrac{z^{-1}}{z^{-1}} = \dfrac{z^{2n+1}-z}{z-z^{-1}}$	A1	Divides numerator and denominator by z. Must see at least $\dfrac{z^{2n+2}-z^2}{z^2-1}$, AG.
		2	
7(b)	$z - z^{-1} = 2\mathrm{i}\sin\theta$	B1	Simplifies denominator.
	$\dfrac{z^{2n+1}-z}{z-z^{-1}} = \dfrac{\cos(2n+1)\theta + \mathrm{i}\sin(2n+1)\theta - \cos\theta - \mathrm{i}\sin\theta}{2\mathrm{i}\sin\theta}$	M1 A1	Applies de Moivre's theorem to numerator.
	$\sum_{r=1}^{n}\cos(2r\theta) = \dfrac{\sin(2n+1)\theta - \sin\theta}{2\sin\theta} = \dfrac{\sin(2n+1)\theta}{2\sin\theta} - \dfrac{1}{2}$	M1	Equates real parts.
	$1 + 2\sum_{r=1}^{n}\cos(2r\theta) = \dfrac{\sin(2n+1)\theta}{\sin\theta}$	A1	AG
		5	

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Q3.

5(a)	$\frac{z-z^{n+1}}{1-z}$ or $\frac{z^{n+1}-z}{z-1}$.	**B1**	
		1	
5(b)	$z^n=1$ and $z\neq1$ leading to $z+z^2+z^3+...+z^n=0$ leading to $1+z+z^2+...+z^{n-1}=0$	**M1 A1**	Must see $z^n=1$.
		2	
5(c)	$\sum_{m=1}^{\infty}z^m=\frac{z}{1-z}=\frac{\cos\theta+\mathrm{i}\sin\theta}{3-\cos\theta-\mathrm{i}\sin\theta}$	**M1 A1**	Applies sum to infinity and substitutes for z.
	$\frac{(\cos\theta+\mathrm{i}\sin\theta)(3-\cos\theta+\mathrm{i}\sin\theta)}{(3-\cos\theta)^2+\sin^2\theta}$	**M1 A1**	Rationalises denominator.
	$\frac{3\cos\theta-\cos^2\theta-\sin^2\theta+\mathrm{i}\sin\theta\cos\theta+\mathrm{i}\sin\theta(3-\cos\theta)}{9-6\cos\theta+\cos^2\theta+\sin^2\theta}$	**M1**	Applies $\sin^2\theta+\cos^2\theta=1$ or $\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}=2\cos\theta$.
	$\mathrm{Re}\left(\sum_{m=1}^{\infty}z^m\right)=\frac{3\cos\theta-1}{10-6\cos\theta}$	**M1 A1**	Takes the real part, AG.
		7	

Q4.

1(a)	$a=1,\quad b=\mathrm{i}$	**B1**	
		1	
1(b)	$z=\mathrm{e}^{\mathrm{i}\frac{2k}{5}\pi}$	**M1**	Finds one fifth root of unity.
	$z=\mathrm{e}^{\mathrm{i}\frac{2k}{5}\pi},\quad k=0,1,2,3,4$	**A1**	Gives all fifth roots of unity. Accept 1 not in exponential form. A0 if $r=1$ not seen or implied.
	Argument of i is $\frac{1}{2}\pi$.	**B1**	
	$z=\mathrm{e}^{\mathrm{i}\frac{1}{6}\pi}$	**M1 A1**	Finds one root of $z^3=\mathrm{i}$. A0 if first root not given in exponential form.
	$z=\mathrm{e}^{\mathrm{i}\frac{5}{6}\pi},\quad \mathrm{e}^{\mathrm{i}\frac{3}{2}\pi}$	**A1**	Finds other two roots of $z^3=\mathrm{i}$. A0 if $r=1$ not seen or implied. Withhold final A1 if a and b reversed in part **(b)** but all other work correct.
		6	**SC** If $z=\mathrm{e}^{\mathrm{i}\left(\frac{1}{10}\pi+\frac{2k}{5}\pi\right)},\quad k=0,1,2,3,4$ award M1 A1 A1 only. **SC** If $z=\mathrm{e}^{\mathrm{i}\left(\frac{2k}{3}\pi\right)},\quad k=0,1,2$ award M1 A1 only.

Complex Numbers 2 MS

Q5.

4	$z-z^{-1}=2\mathrm{i}\sin\theta$ and $z+z^{-1}=2\cos\theta$	**B1**	Use of $z-z^{-1}=2\mathrm{i}\sin\theta$ and $z+z^{-1}=2\cos\theta$
	$\left(z-z^{-1}\right)^5=\left(z^5-z^{-5}\right)-5\left(z^3-z^{-3}\right)+10\left(z-z^{-1}\right)$	**M1 A1**	Expands and groups. A0 if grouping not shown.
	$\left(z+z^{-1}\right)^5=\left(z^5+z^{-5}\right)+5\left(z^3+z^{-3}\right)+10\left(z+z^{-1}\right)$	**M1 A1**	Expands and groups. A0 if grouping not shown.
	$\frac{2^5\mathrm{i}\sin^5\theta}{2^5\cos^5\theta}=\frac{2\mathrm{i}\sin 5\theta-5(2\mathrm{i}\sin 3\theta)+10(2\mathrm{i}\sin\theta)}{2\cos 5\theta+5(2\cos 3\theta)+10(2\cos\theta)}$	**M1**	Substitutes $z^n+z^{-n}=2\cos n\theta$ and $z^n-z^{-n}=2\mathrm{i}\sin n\theta$.
	$\tan^5\theta=\frac{\sin 5\theta-5\sin 3\theta+10\sin\theta}{\cos 5\theta+5\cos 3\theta+10\cos\theta}$.	**A1**	Justifies cancellation of constants.
		7	

Q6.

6(a)	$(\cos 5\theta+\mathrm{i}\sin 5\theta)=(c+\mathrm{i}s)^5$	**M1**	Uses binomial theorem.
	$\sin 5\theta=s^5-10c^2s^3+5c^4s$	**A1**	
	$s^5-10\left(1-s^2\right)s^3+5\left(1-s^2\right)^2s$	**M1**	Uses $c^2=1-s^2$ or $\cot^2\theta=\operatorname{cosec}^2\theta-1$ after dividing numerator and denominator by s^5.
	$16s^5-20s^3+5s$	**A1**	
	$\operatorname{cosec}5\theta=\frac{1}{16s^5-20s^3+5s}\times\frac{\operatorname{cosec}^5\theta}{\operatorname{cosec}^5\theta}$	**M1**	Divides *simplified* numerator and denominator by s^5.
	$\frac{\operatorname{cosec}^5\theta}{5\operatorname{cosec}^4\theta-20\operatorname{cosec}^2\theta+16}$	**A1**	AG
		6	
6(b)	$x^5=2\left(5x^4-20x^2+16\right)$ leading to $\frac{x^5}{5x^4-20x^2+16}=2$	**M1**	Relates with equation in part (a).
	$\operatorname{cosec}5\theta=2$ leading to $\sin 5\theta=\frac{1}{2}$	**M1**	Solves $\sin 5\theta=\frac{1}{2}$.
	$x=\operatorname{cosec}\left(\frac{1}{30}\pi\right)$	**A1**	Gives one correct solution.
	$\operatorname{cosec}\left(\frac{5}{30}\pi\right),\operatorname{cosec}\left(-\frac{7}{30}\pi\right),\operatorname{cosec}\left(-\frac{11}{30}\pi\right),\operatorname{cosec}\left(\frac{13}{30}\pi\right)$	**A1**	Gives four other solutions. Allow different values of q as long as all five solutions are found. 9.56, 2, –1.49, –1.09, 1.02
		4	

Q7.

4(a)	$\mathrm{e}^{\mathrm{i}\frac{2}{5}k\pi}$, $k=0,1,2,3,4$	**B2**	OE. B1 for 1 correct fifth root of unity. B2 for exactly 5 distinct, correct roots.
		2	
4(b)	$(c+\mathrm{i}s)^4=c^4+4c^3(\mathrm{i}s)+6c^2(\mathrm{i}s)^2+4c(\mathrm{i}s)^3+(\mathrm{i}s)^4$	**M1 A1**	Uses binomial expansion. Can also go RHS to LHS using $2\cos\theta=z+z^{-1}$.
	$\cos 4\theta=c^4-6c^2s^2+s^4=c^4-6c^2(1-c^2)+(1-c^2)^2$	**M1**	Applies $s^2=1-c^2$ or $2\cos\theta=z+z^{-1}$.
	$8\cos^4\theta-8\cos^2\theta+1$	**A1**	AG. Can get final A1 without first. **SC:** Using double angle formulae scores 1/4.
		4	

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4(c)	$8x^9-8x^7+x^5-8x^4+8x^2-1=(x^5-1)(8x^4-8x^2+1)$	**B1**	
	$\cos 4\theta=0$ leading to $4\theta=\frac{1}{2}(2k+1)\pi$	**M1 A1**	Solves $\cos 4\theta=0$. A1 for $4\theta=\frac{1}{2}\pi,\frac{3}{2}\pi,\frac{5}{2}\pi,\frac{7}{2}\pi$ OE.
	$\cos 0,\ \cos\frac{1}{8}\pi,\ \cos\frac{3}{8}\pi,\ \cos\frac{5}{8}\pi,\ \cos\frac{7}{8}\pi$	**A1**	Gives exactly 5 distinct, real roots. Accept $1, \pm\cos\frac{1}{8}\pi, \pm\cos\frac{3}{8}\pi$.
		4	

Q8.

1	$z^3=7\sqrt{3}-7\mathrm{i}=14\mathrm{e}^{-\mathrm{i}\frac{1}{6}\pi}$	**B1**	Converts $7\sqrt{3}-7\mathrm{i}$ to exponential form.
	$z_1=14^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{1}{18}\pi}$	**M1 A1**	Finds one root.
	$z_2=14^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{11}{18}\pi},\ z_3=14^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{13}{18}\pi}$	**A1FT A1FT**	Finds other two roots. FT on modulus. Others given scores A1 A0.
		5	

Q9.

8(c)	$z+z^{-1}=2\cos\theta$	**B1**	
	$\left(z+z^{-1}\right)^5=\left(z^5+z^{-5}\right)+5\left(z^3+z^{-3}\right)+10\left(z+z^{-1}\right)$	**M1 A1**	Expands and groups. A0 if no grouping shown.
	$2^5\cos^5\theta=2\cos 5\theta+5(2\cos 3\theta)+10(2\cos\theta)$	**M1**	Substitutes $z^n+z^{-n}=2\cos n\theta$.
	$\cos^5\theta=\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta$	**A1**	
		5	

Q10.

7(a)	$\frac{w^n-1}{w-1}$	**B1**	
		1	
7(b)	$(1+\mathrm{i}\tan\theta)^k=\sec^k\theta(\cos\theta+\mathrm{i}\sin\theta)^k=\sec^k\theta(\cos k\theta+\mathrm{i}\sin k\theta)$	**M1 A1**	Applies de Moivre's theorem, AG.
		2	
7(c)	$\sum_{k=0}^{n-1}(1+\mathrm{i}\tan\theta)^k=\frac{(1+\mathrm{i}\tan\theta)^n-1}{\mathrm{i}\tan\theta}$	**M1 A1**	Applies part **(a)**.
	$-\cot\theta\sec^n\theta(\mathrm{i}\cos n\theta-\sin n\theta)+\mathrm{i}\cot\theta$	**A1**	
	$\sum_{k=0}^{n-1}\sec^k\theta\sin k\theta=\cot\theta\left(1-\sec^n\theta\cos n\theta\right)$	**M1 A1**	Takes imaginary part, AG.
		5	
7(d)	$\theta=\frac{1}{3}\pi\quad\sum_{k=0}^{6m-1}2^k\sin\left(\frac{1}{3}k\pi\right)=\frac{1}{3}\sqrt{3}\left(1-2^{6m}\cos(2m\pi)\right)=\frac{1}{3}\sqrt{3}\left(1-2^{6m}\right)$	**M1 A1**	Sets $\theta=\frac{1}{3}\pi$.
		2	