Complex Numbers 2

Q1.

(a) Use de Moivre's theorem to show that $\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$. [6]

It is given that $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$.

(b) Find the exact value of $\int_0^{\frac{1}{3}\pi}\left(\cos^6(\frac{1}{4}x) + \sin^6(\frac{1}{4}x)\right)dx$. [4]

(c) Express each root of the equation $16c^6 + 16(1-c^2)^3 - 13 = 0$ in the form $\cos k\pi$, where k is a rational number. [5]

Q2.

(a) Show that $\sum_{r=1}^{n} z^{2r} = \frac{z^{2n+1} - z}{z - z^{-1}}$, for $z \neq 0, 1, -1$. [2]

(b) By letting $z = \cos\theta + i\sin\theta$, show that, if $\sin\theta \neq 0$,

$$1 + 2\sum_{r=1}^{n}\cos(2r\theta) = \frac{\sin(2n+1)\theta}{\sin\theta}.$$ [5]

Q3.

(a) State the sum of the series $z + z^2 + z^3 + ... + z^n$, for $z \neq 1$. [1]

(b) Given that z is an nth root of unity and $z \neq 1$, deduce that $1 + z + z^2 + ... + z^{n-1} = 0$. [2]

(c) Given instead that $z = \frac{1}{3}(\cos\theta + i\sin\theta)$, use de Moivre's theorem to show that

$$\sum_{m=1}^{\infty} 3^{-m}\cos m\theta = \frac{3\cos\theta - 1}{10 - 6\cos\theta}.$$ [7]

Q4.

(a) Find a and b such that

$$z^8 - iz^5 - z^3 + i = (z^5 - a)(z^3 - b).$$ [1]

(b) Hence find the roots of

$$z^8 - iz^5 - z^3 + i = 0,$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [6]

Complex Numbers 2

Q5.

By considering the binomial expansions of $\left(z+\frac{1}{z}\right)^5$ and $\left(z-\frac{1}{z}\right)^5$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\tan^5\theta = \frac{\sin 5\theta - a\sin 3\theta + b\sin\theta}{\cos 5\theta + a\cos 3\theta + b\cos\theta},$$

where a and b are integers to be determined. [7]

Q6.

(a) Use de Moivre's theorem to show that

$$\operatorname{cosec} 5\theta = \frac{\operatorname{cosec}^5\theta}{5\operatorname{cosec}^4\theta - 20\operatorname{cosec}^2\theta + 16}.$$ [6]

(b) Hence obtain the roots of the equation

$$x^5 - 10x^4 + 40x^2 - 32 = 0$$

in the form $\operatorname{cosec}(q\pi)$, where q is rational. [4]

Q7.

(a) Write down all the roots of the equation $x^5 - 1 = 0$. [2]

(b) Use de Moivre's theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. [4]

(c) Use the results of parts **(a)** and **(b)** to express each real root of the equation

$$8x^9 - 8x^7 + x^5 - 8x^4 + 8x^2 - 1 = 0$$

in the form $\cos k\pi$, where k is a rational number. [4]

Q8.

Find the roots of the equation $z^3 = 7\sqrt{3} - 7\mathrm{i}$, giving your answers in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5]

Complex Numbers 2

Q9.

(c) By considering the binomial expansion of $\left(z+\frac{1}{z}\right)^5$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cos^5\theta = a\cos 5\theta + b\cos 3\theta + c\cos\theta,$$

where a, b and c are constants to be determined. [5]

Q10.

(a) State the sum of the series $1+w+w^2+w^3+\ldots+w^{n-1}$, for $w \neq 1$. [1]

(b) Show that $(1+\mathrm{i}\tan\theta)^k = \sec^k\theta(\cos k\theta + \mathrm{i}\sin k\theta)$, where θ is not an integer multiple of $\frac{1}{2}\pi$. [2]

(c) By considering $\sum_{k=0}^{n-1}(1+\mathrm{i}\tan\theta)^k$, show that

$$\sum_{k=0}^{n-1}\sec^k\theta\sin k\theta = \cot\theta(1-\sec^n\theta\cos n\theta),$$

provided θ is not an integer multiple of $\frac{1}{2}\pi$. [5]

(d) Hence find $\sum_{k=0}^{6m-1}2^k\sin\left(\frac{1}{3}k\pi\right)$ in terms of m. [2]
