Integration 2 MS

Q1.

5(i)	$\dot{x}^2+\dot{y}^2=\left(t^{\frac{3}{2}}-t^{-\frac{1}{2}}\right)^2+4t=\ldots=\left(t^{\frac{3}{2}}+t^{-\frac{1}{2}}\right)^2$	M1A1	SOI
	$s=\int_1^4\left(t^{\frac{3}{2}}+t^{-\frac{1}{2}}\right)\mathrm{d}t$	M1	
	$=\left[\frac{2}{5}t^{\frac{5}{2}}+2t^{\frac{1}{2}}\right]_1^4$	M1	
	$=\left[\frac{64}{5}+4\right]-\left[\frac{2}{5}+2\right]\quad=\frac{72}{5}$	A1	
	Total:	**5**	
5(ii)	$S=2\pi\int_1^4\frac{4}{3}t^{\frac{3}{2}}\left(t^{\frac{3}{2}}+t^{-\frac{1}{2}}\right)\mathrm{d}t=\frac{8}{3}\pi\int_1^4(t^3+t)\mathrm{d}t$	*M1	
	$=\left(\frac{8}{3}\pi\right)\left[\frac{1}{4}t^4+\frac{1}{2}t^2\right]_1^4$	DM1	
	$=\frac{8}{3}\pi\left\{[64+8]-\left[\frac{1}{4}+\frac{1}{2}\right]\right\}\quad=\ 190\pi$	A1	
	Total:	**3**	

Q2.

8(i)	$I_2=\int_0^{\frac{1}{4}\pi}\sec^2x\mathrm{d}x=[\tan x]_0^{\frac{1}{4}\pi}=1$	M1A1	
		2	
8(ii)	$I_n=\int_0^{\frac{1}{4}\pi}\sec^{n-2}x.\sec^2x\mathrm{d}x$	M1	
	$=\left[\sec^{n-2}x\tan x\right]_0^{\frac{1}{4}\pi}-\int_0^{\frac{1}{4}\pi}(n-2)\sec^{n-3}x(\sec x\tan x)\tan x\mathrm{d}x$	M1A1	
	$=\left[\sec^{n-2}x\tan x\right]_0^{\frac{1}{4}\pi}-(n-2)\int_0^{\frac{1}{4}\pi}\sec^{n-2}x\left(\sec^2x-1\right)\mathrm{d}x$	M1A1	
	$\Rightarrow(n-1)I_n=2^{\frac{1}{2}n-1}+(n-2)I_{n-2}$		AG
		5	

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Q3.

1	$\frac{dx}{dt} = e^t - 1$ and $\frac{dy}{dt} = 2e^{\frac{1}{2}t}$	B1	
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(e^t + 1\right)^2$	M1 A1	M1 for using $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$
	Arc length is $\int_0^3 e^t + 1 \, dt = \left[e^t + t\right]_0^3$	M1	M1 for good attempt at correct integral
	$= 2 + e^3$ (or 22.1)	A1	
		5	

Q4.

2	$3 + 4x - 4x^2 = 4 - (2x-1)^2$	1	M1	Complete the square
	EITHER Solution 1 $2x - 1 = 2\sin\theta$: $\sqrt{4 - 4\sin^2\theta} = 2\cos\theta$:	1	(M1	Use appropriate substitution
	Integral = $\int \frac{\cos\theta}{2\cos\theta} d\theta = \frac{1}{2}\theta = \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{2}\right)$	2	M1A1)	Integrate
	OR Solution 2 $\frac{1}{2}\int \frac{1}{\sqrt{1 - \left(x - \frac{1}{2}\right)^2}} dx$	1	(M1	Use formula
	$= \frac{1}{2}\sin^{-1}\left(x - \frac{1}{2}\right)$	2	M1A1)	
	$\frac{1}{2}\left(\sin^{-1}(0.5) - \sin^{-1}(-0.5)\right)$	1	M1	Use limits
	$= \frac{\pi}{6}$	1	A1	CAO NOTE: An answer of $\frac{\pi}{6}$ without working scores 0 (as per front cover instructions)
	Available marks	6		

Q5.

4(a)	$\int_0^1 x^2 \, dx < \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$	M1 A1
	$\frac{1}{n^3}\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{2n^2+3n+1}{6n^2}$	M1 A1
		4
4(b)	$\int_0^1 x^2 \, dx > \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2$	M1 A1
	$= \frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = \frac{(n-1)(n)(2n-2+1)}{6n^3} = \frac{2n^2-3n+1}{6n^2}$	M1 A1
		4

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Q6.

6(a)	$I_1 = \int_0^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} \mathrm{d}x = \left[\sin^{-1} x\right]_0^{\frac{1}{2}} = \frac{1}{6}\pi$	**M1 A1**
		2
6(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x^2)^{-\frac{1}{2}n}\right) = nx^2(1-x^2)^{-\frac{1}{2}n-1} + (1-x^2)^{-\frac{1}{2}n}$	**M1 A1**
	$= n\left(1-(1-x^2)\right)(1-x^2)^{-\frac{1}{2}n-1} + (1-x^2)^{-\frac{1}{2}n}$	**M1**
	$\left[x(1-x^2)^{-\frac{1}{2}n}\right]_0^{\frac{1}{2}} = nI_{n+2} - nI_n + I_n$	**M1**
	$\frac{1}{2}\left(\frac{3}{4}\right)^{-\frac{1}{2}n} = nI_{n+2} - (n-1)I_n \Rightarrow nI_{n+2} = 2^{n-1}3^{-\frac{1}{2}n} + (n-1)I_n$	**A1**
		5
6(c)	$I_3 = 3^{-\frac{1}{2}}$	**B1**
	$3I_5 = 2^2 3^{-\frac{3}{2}} + 2I_3 \Rightarrow I_5 = \frac{10}{27}\sqrt{3}$	**M1 A1**
		3

Q7.

2	$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\sinh^2 x} = \cosh x$	**M1 A1**	Applies $\cosh^2 x = 1+\sinh^2 x$
	$2\pi\int_0^{\frac{1}{2}} \cosh^2 x \,\mathrm{d}x$	**M1**	Correct formula, correct limits.
	$= \pi\int_0^{\frac{1}{2}} \cosh 2x + 1 \,\mathrm{d}x$	**M1**	Applies $2\cosh^2 x = \cosh 2x + 1$ or expands $2\left(e^x + e^{-x}\right)^2$
	$= \pi\left[\frac{1}{2}\sinh 2x + x\right]_0^{\frac{1}{2}}$	**A1**	Correct integration.
	$= \frac{1}{4}\pi\left(e - e^{-1} + 2\right)$	**A1**	
		6	

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Q8.

8(a)	$\frac{dx}{dt}=2\sinh t$	B1	
	$\frac{dy}{dt}=\frac{3}{2}-\frac{1}{2}\cosh 2t=1-\left(\frac{1}{2}\cosh 2t-\frac{1}{2}\right)=1-\sinh^2 t$	M1 A1	Applies $2\sinh^2 t=\cosh 2t-1$, AG.
		3	
8(b)(i)	$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=4s^2+(s^2-1)^2=4s^2+s^4-2s^2+1=(s^2+1)^2$	M1	Factorises $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$.
	$\cosh^4 t$	A1	
	$2\pi\int_0^1\left(\frac{3}{2}t-\frac{1}{4}\sinh 2t\right)\cosh^2 t\ dt=\pi\int_0^1\left(\frac{3}{2}t-\frac{1}{4}\sinh 2t\right)(\cosh 2t+1)dt$	M1 A1	Correct formula for surface area, AG. A0 if limits missing. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$ does not need to be simplified for M1.
		4	

8(b)(ii)	$\int_0^1\frac{1}{4}\sinh 2t(\cosh 2t+1)\,dt=\left[\frac{1}{16}(1+\cosh 2t)^2\right]_0^1=\frac{1}{16}(1+\cosh 2)^2-\frac{1}{4}$	M1 A1	Integrates.
	$\frac{3}{2}\int_0^1 t(\cosh 2t+1)dt=\frac{3}{2}\left[t\left(\frac{1}{2}\sinh 2t+t\right)\right]_0^1-\frac{3}{2}\int_0^1\frac{1}{2}\sinh 2t+t\ dt$	M1 A1	Integrates by parts.
	$\frac{3}{2}\left[t\left(\frac{1}{2}\sinh 2t+t\right)-\frac{1}{4}\cosh 2t-\frac{1}{2}t^2\right]_0^1=\frac{3}{2}\left(\frac{1}{2}\sinh 2-\frac{1}{4}\cosh 2+\frac{3}{4}\right)$	A1	
	$\pi\left(\frac{3}{4}\sinh 2-\frac{3}{8}\cosh 2-\frac{1}{16}(1+\cosh 2)^2+\frac{11}{8}\right)$	A1	OE. Must be exact. (Decimal answer is 3.980131435…)
		6	

Q9.

7(a)	$I_1=\int_0^{\frac{3}{2}}\left(4+x^2\right)^{-\frac{1}{2}}=\left[\sinh^{-1}\left(\frac{1}{2}x\right)\right]_0^{\frac{3}{2}}$	M1 A1	Recognises integral or uses appropriate substitution.
	$\ln 2$	A1	Insert limits, must simplify.
		3	
7(b)	$\frac{d}{dx}\left(x\left(4+x^2\right)^{-\frac{1}{2}n}\right)=-nx^2\left(4+x^2\right)^{-\frac{1}{2}n-1}+\left(4+x^2\right)^{-\frac{1}{2}n}$	M1 A1	Uses the product rule to differentiate.
	$-n\left(4+x^2-4\right)\left(4+x^2\right)^{-\frac{1}{2}n-1}+\left(4+x^2\right)^{-\frac{1}{2}n}$	M1	Applies $x^2=4+x^2-4$.
	$\left[x\left(4+x^2\right)^{-\frac{1}{2}n}\right]_0^{\frac{3}{2}}=-nI_n+4nI_{n+2}+I_n$	M1	Integrates both sides using the limits given.
	$\frac{3}{2}\left(\frac{2}{5}\right)^n=(1-n)I_n+4nI_{n+2}\Rightarrow 4nI_{n+2}=\frac{3}{2}\left(\frac{2}{5}\right)^n+(n-1)I_n$	A1	Substitutes limits and rearranges, AG.
	Alternative method for question 7(b)		
	$I_n=\left[x\left(4+x^2\right)^{-\frac{1}{2}n}\right]_0^{\frac{3}{2}}+n\int_0^{\frac{3}{2}}x^2\left(4+x^2\right)^{-\frac{1}{2}n-1}dx$	M1 A1	Integrates by parts.
	$I_n=\left[x\left(4+x^2\right)^{-\frac{1}{2}n}\right]_0^{\frac{3}{2}}+n\int_0^{\frac{3}{2}}\left(4+x^2-4\right)\left(4+x^2\right)^{-\frac{1}{2}n-1}dx$	M1	Applies $x^2=4+x^2-4$.
	$I_n=\frac{3}{2}\left(\frac{2}{5}\right)^n+nI_n-4I_{n+2}\Rightarrow 4nI_{n+2}=\frac{3}{2}\left(\frac{2}{5}\right)^n+(n-1)I_n$	M1 A1	Substitutes limits and rearranges, AG.
		5	

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7(c)	$I_3 = \frac{3}{20}$	**B1**	Applies reduction formula with $n = 1$.
	$12I_5 = \frac{3}{2}\left(\frac{8}{125}\right) + 2I_3 \Rightarrow I_5 = \frac{33}{1000} = 0.033$	**M1 A1**	Applies reduction formula with $n = 3$.
		3	

Q10.

4(a)	$\sum_{r=1}^{N} \frac{\ln r}{r^2} = \frac{\ln 2}{4} + \sum_{r=3}^{N} \frac{\ln r}{r^2}$	**M1 A1**	Compares with sum of the areas of the rectangles. M1 for writing out sum, A1 for considering $\sum_{r=3}^{N} \frac{\ln r}{r^2}$.
	$< \frac{\ln 2}{4} + \int_2^N \frac{\ln x}{x^2} \mathrm{d}x$	**M1**	Compares with integral.
	$\int_2^N \frac{\ln x}{x^2} \mathrm{d}x = \left[-\frac{\ln x + 1}{x} \right]_2^N$	**M1 A1**	Finds integral.
	$\sum_{r=1}^{N} \frac{\ln r}{r^2} < \frac{\ln 2}{4} + \left(-\frac{\ln N + 1}{N} + \frac{\ln 2 + 1}{2} \right) = \frac{2 + 3\ln 2}{4} - \frac{1 + \ln N}{N}$	**M1 A1**	Inserts limits, AG. A1 requires second M1.
		7	
4(b)	$\sum_{r=1}^{N} \frac{\ln r}{r^2} = \sum_{r=2}^{N-1} \frac{\ln r}{r^2} + \frac{\ln N}{N^2} > \int_2^N \frac{\ln x}{x^2} \mathrm{d}x + \frac{\ln N}{N^2}$	**M1 A1**	Compares with integral. Lower limit of 1 scores M0. Accept $\sum_{r=1}^{N} \frac{\ln r}{r^2} > \int_2^{N+1} \frac{\ln x}{x^2}$.
	$= \frac{\ln 2 + 1}{2} - \frac{\ln N + 1}{N} + \frac{\ln N}{N^2}$	**A1**	Accept $\frac{\ln 2 + 1}{2} - \frac{\ln(N+1) + 1}{N+1}$.
		3	