Integration 2

Q1.

A curve C has parametric equations

$$x = \tfrac{2}{5}t^{\frac{5}{2}} - 2t^{\frac{1}{2}}, \quad y = \tfrac{4}{3}t^{\frac{3}{2}}, \quad \text{for } 1 \leqslant t \leqslant 4.$$

(i) Find the exact value of the arc length of C. [5]

(ii) Find also the exact value of the surface area generated when C is rotated through 2π radians about the x-axis. [3]

Q2.

Let $I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \, dx$ for $n > 0$.

(i) Find the value of I_2. [2]

(ii) Show that, for $n > 2$,

$$(n-1)I_n = 2^{\frac{1}{2}n-1} + (n-2)I_{n-2}. \quad [5]$$

(iii) The curve C has equation $y = \sec^3 x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The region R is bounded by C, the x-axis, the y-axis and the line $x = \frac{1}{4}\pi$. Find the volume of revolution generated when R is rotated through 2π radians about the x-axis. [4]

Q3.

The curve C is defined parametrically by

$$x = e^t - t, \quad y = 4e^{\frac{1}{2}t}.$$

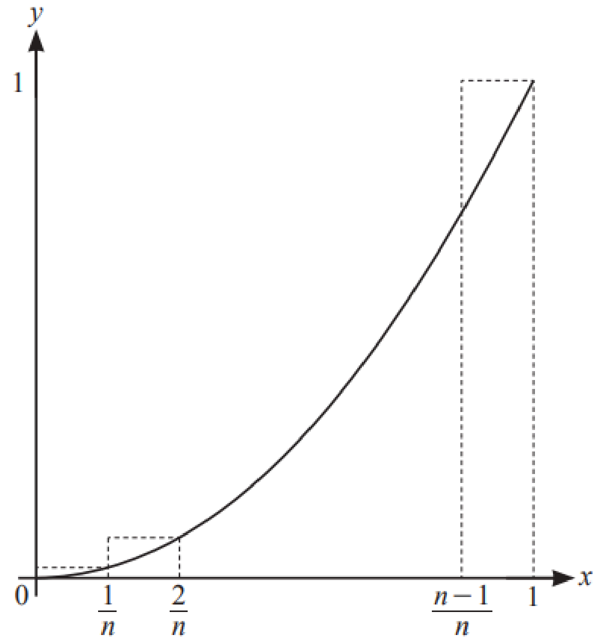
Find the length of the arc of C from the point where $t = 0$ to the point where $t = 3$. [5]

Q4.

Find the exact value of $\int_0^1 \frac{1}{\sqrt{3 + 4x - 4x^2}} \, dx$. [6]

Integration 2

Q5.



The diagram shows the curve with equation $y = x^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 x^2 \, dx < \frac{2n^2 + 3n + 1}{6n^2}.$$ [4]

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 x^2 \, dx$. [4]

Q6.

The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{1}{2}} \left(1 - x^2\right)^{-\frac{1}{2}n} \, dx$.

(a) Find the exact value of I_1. [2]

(b) By considering $\frac{d}{dx}\left(x\left(1 - x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$nI_{n+2} = 2^{n-1} 3^{-\frac{1}{2}n} + (n-1)I_n.$$ [5]

(c) Find the exact value of I_5 giving the answer in the form $k\sqrt{3}$, where k is a rational number to be determined. [3]

Integration 2

Q7.

A curve has equation $y = \cosh x$, for $0 \leqslant x \leqslant \frac{1}{2}$.

Find, in terms of π and e, the area of the surface generated when the curve is rotated through 2π radians about the x-axis. [6]

Q8.

The curve C has parametric equations

$$x = 2\cosh t, \quad y = \tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t, \quad \text{for } 0 \leqslant t \leqslant 1.$$

(a) Find $\frac{dx}{dt}$ and show that $\frac{dy}{dt} = 1 - \sinh^2 t$. [3]

The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by A.

(b) **(i)** Show that $A = \pi \int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,dt$. [4]

(ii) Hence find A in terms of π, $\sinh 2$ and $\cosh 2$. [6]

Q9.

The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{3}{2}} (4 + x^2)^{-\frac{1}{2}n}\,dx$.

(a) Find the exact value of I_1, expressing your answer in logarithmic form. [3]

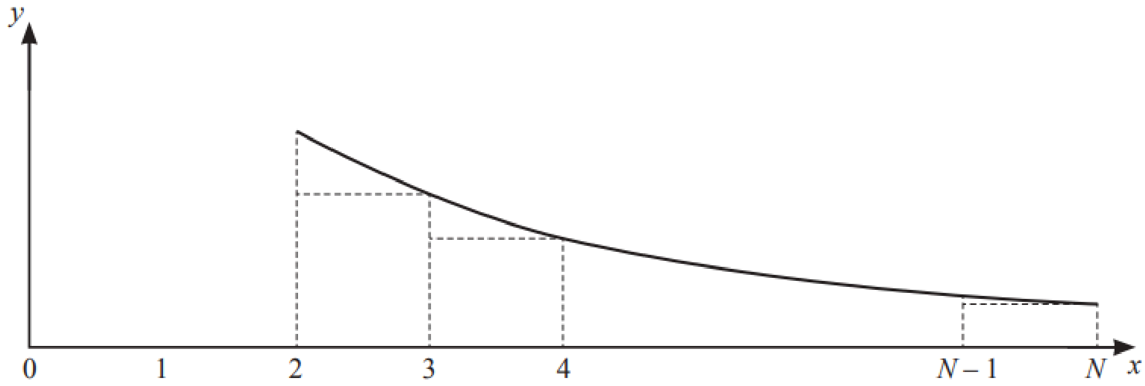
(b) By considering $\frac{d}{dx}\left(x(4 + x^2)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$4nI_{n+2} = \tfrac{3}{2}\left(\tfrac{2}{5}\right)^n + (n-1)I_n.$$ [5]

(c) Find the value of I_5. [3]

Integration 2

Q10.



The diagram shows the curve with equation $y = \frac{\ln x}{x^2}$ for $x \geqslant 2$, together with a set of $(N-2)$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{\ln r}{r^2} < \frac{2+3\ln 2}{4} - \frac{1+\ln N}{N}.$$ [7]

(b) Use a similar method to find, in terms of N, a lower bound for $\sum_{r=1}^{N} \frac{\ln r}{r^2}$. [3]
