Matrices 1 MS

Q1.

11		**EITHER**				
	(i)	Writes **P** and **D**. (Note: Columns can be in any order, but must match.)	$\mathbf{P} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	B1B1		
		Finds Det **P.**	Det **P** $= 2$	B1		
		Finds inverse of **P**. (Adj ÷ Det)	$\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (No working 1/3) Row operations M1A1A1 (3 errors).	M1A1		
		Finds expression for **A.**	$\mathbf{A} = \mathbf{PDP}^{-1}$	M1		
		Evaluates **A.**	$\mathbf{A} = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} . \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	M1A1		
			$= \begin{pmatrix} 1.5 & 0.5 & 0.5 \\ 1.5 & 0.5 & 1.5 \\ -1 & 1 & 0 \end{pmatrix}$	A1	9	
	(ii)	Finds expression for $\mathbf{A}^{2n}$	$\mathbf{A}^{2n} = \mathbf{PD}^{2n}\mathbf{P}^{-1}$	M1		
			$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{2n} \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	A1		
			$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 2^{2n} \\ 1 & 0 & 2^{2n} \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	M1A1		
		Evaluates.	$= \frac{1}{2}\begin{pmatrix} 2^{2n}+1 & 2^{2n}-1 & 2^{2n}-1 \\ 2^{2n}-1 & 2^{2n}+1 & 2^{2n}-1 \\ 0 & 0 & 2 \end{pmatrix}$	A1	5	**[14]**

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Q2.

5	Identifies matrices **P** and **D.**	$\mathbf{P} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -3 \\ -2 & 3 & 5 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	B1B1		
	Finds inverse of **P**.	Det **P** =1	B1		
		$\mathbf{P}^{-1} = \text{Adj } \mathbf{P} = \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$	M1A1		
	Uses appropriate result to obtain **A**. (First mark can be implied by correct working.)	$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} \Rightarrow \mathbf{A} = \mathbf{PDP}^{-1}$	M1		
		$\mathbf{A} = \begin{pmatrix} 0 & -1 & 2 \\ -1 & -1 & -3 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$			
		$= \begin{pmatrix} 0 & -1 & 4 \\ -1 & -1 & -6 \\ 2 & 3 & 10 \end{pmatrix}\begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$	M1A1√		
		$= \begin{pmatrix} 3 & 4 & 2 \\ -11 & -27 & -13 \\ 21 & 54 & 26 \end{pmatrix}$	A1	9	**[9]**
5	**Alternative Approach:** $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	Use of $\mathbf{Ae} = \lambda\mathbf{e}$	(M1)		
		Obtains 3 sets of 3 linear equations: One set Other two sets	 (M1A1) (A1A1)		
		Solves one set Solves other sets	(M1A1) (A1A1)	(9)	**[9]**

Q3.

6	Shows **e** is an eigenvector of **A** and gives eigenvalue.	$\mathbf{Ae} = 2\mathbf{e} \Rightarrow$ **e** is an eigenvector with eigenvalue 2.	M1A1	2	
	Finds characteristic equation.	$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$	M1A1		
	Factorises.	$\Rightarrow (\lambda - 2)(\lambda^2 - 1) = 0$	A1		
	States other eigenvalues.	Other eigenvalues are –1 and 1.	A1	4	
	Repeats for **B**.	$\mathbf{Be} = 3\mathbf{e} \Rightarrow$ **e** is an eigenvector with eigenvalue 3	B1		
	States result for **AB**.	**ABe = A.3e = 3Ae = 3.2e = 6e** **AB** has eigenvector **e** with eigenvalue 6	M1A1	3	**[9]**

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Q4.

6	$\lambda=1:\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 10 & -8 & 10\\ 7 & -5 & 7\end{vmatrix}=\begin{pmatrix}-6\\0\\6\end{pmatrix}\sim\begin{pmatrix}1\\0\\-1\end{pmatrix}$ oe	**M1A1**
	$\lambda=3:\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & 0 & 0\\ 10 & -10 & 10\end{vmatrix}=\begin{pmatrix}0\\20\\20\end{pmatrix}\sim\begin{pmatrix}0\\1\\1\end{pmatrix}$ oe	**A1** [3]
	$\begin{pmatrix}1 & 0 & 0\\ 10 & -7 & 10\\ 7 & -5 & 8\end{pmatrix}\begin{pmatrix}0\\2\\1\end{pmatrix}=\begin{pmatrix}0\\-4\\-2\end{pmatrix}=-2\begin{pmatrix}0\\2\\1\end{pmatrix}\Rightarrow\lambda=-2$	**M1A1** [2]
	$\mathbf{D}=\begin{pmatrix}-2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{pmatrix}$, $\mathbf{P}=\begin{pmatrix}0 & 1 & 0\\ 2 & 0 & 1\\ 1 & -1 & 1\end{pmatrix}$ (or other multiples or permutations).	**B1√ B1√**
	Det $\mathbf{P}=-1$ (or 1 depending on permutation).	**B1**
	Adj $\mathbf{P}=\begin{pmatrix}1 & -1 & 1\\ -1 & 0 & 0\\ -2 & 1 & 2\end{pmatrix}\Rightarrow\mathbf{P}^{-1}=\begin{pmatrix}-1 & 1 & -1\\ 1 & 0 & 0\\ 2 & -1 & 2\end{pmatrix}$ (or other permutations).	**M1A1** [5] **Total 10**

Q5.

11E(i)	$\mathbf{Ae}=\lambda\mathbf{e}$ and $\mathbf{Be}=\mu\mathbf{e}$	**M1A1**	
	$\mathbf{ABe}=\mathbf{A}\mu\mathbf{e}=\mu\mathbf{Ae}=\mu\lambda\mathbf{e}=\lambda\mu\mathbf{e}$	**M1**	AG
		3	
11E(ii)	$(\lambda+1)(\lambda^2-5\lambda+6)=0$	**A1**	
	$(\lambda+1)(\lambda-2)(\lambda-3)=0$	**A1**	
	$\lambda=-1,2,3.$	**M1**	
	Eigenvectors are $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$ respectively.	**A1A1**	Uses either vector product or equations to find eigenvectors
		6	
11E(iii)	$\begin{pmatrix}3 & 6 & 1\\ 1 & -2 & -1\\ 6 & 6 & -2\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}-3\\3\\0\end{pmatrix}\Rightarrow\mu_1=-3$	**M1**	
	Similarly, other two eigenvalues of **B** are – 2 and 4.	**A1**	
	Eigenvalues of **AB** are 3 , – 4 and 12	**A1**	
	Corresponding eigenvectors are $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$.	**A1**	
		4	

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Q6.

8(i)	Eigenvalues of (upper diagonal matrix) **A** are $2, m$ and 1. (Or from characteristic equation: $(\lambda-2)(\lambda-m)(\lambda-1)=0$)	B1	
	$\lambda=2:\ \mathbf{e}_1=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & m-2 & 1\\ 0 & 0 & -1\end{vmatrix}=\begin{pmatrix}2-m\\0\\0\end{pmatrix}=t\begin{pmatrix}1\\0\\0\end{pmatrix}$	M1 A1	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda=m:\ \mathbf{e}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2-m & m & 1\\ 0 & 0 & 7\end{vmatrix}=\begin{pmatrix}7m\\7(m-2)\\0\end{pmatrix}=t\begin{pmatrix}m\\m-2\\0\end{pmatrix}$	A1	
	$\lambda=1:\ \mathbf{e}_3=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & m & 1\\ 0 & m-1 & 7\end{vmatrix}=\begin{pmatrix}6m+1\\-7\\m-1\end{pmatrix}=t\begin{pmatrix}6m+1\\-7\\m-1\end{pmatrix}$	A1	
	Thus $\mathbf{P}=\begin{pmatrix}1 & m & 6m+1\\ 0 & m-2 & -7\\ 0 & 0 & m-1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}2 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & 1\end{pmatrix}$	M1 A1 FT	Or correctly matched permutations of columns. No follow through on two or more zero eigenvectors.
		7	

8(ii)	$\mathbf{M}^7\mathbf{P}=\mathbf{P}\mathbf{D}^7\mathbf{P}^{-1}\mathbf{P}=\mathbf{P}\mathbf{D}^7=\begin{pmatrix}1 & m & 6m+1\\ 0 & m-2 & -7\\ 0 & 0 & m-1\end{pmatrix}\begin{pmatrix}2^7 & 0 & 0\\ 0 & m^7 & 0\\ 0 & 0 & 1\end{pmatrix}$	M1 A1 FT	Applies $\mathbf{M}^7=\mathbf{P}\mathbf{D}^7\mathbf{P}^{-1}$.
	$=\begin{pmatrix}2^7 & m^8 & 6m+1\\ 0 & m^8-2m^7 & -7\\ 0 & 0 & m-1\end{pmatrix}$	A1	Order of columns might be swapped depending on **P**.
		3	

Q7.

8(a)	$\begin{vmatrix}3 & 1 & 1\\ a & 6 & -1\\ 0 & a & -2\end{vmatrix}=0\Rightarrow 3(-12+a)-(-2a)+a^2=0$	M1
	$a^2+5a-36=0\Rightarrow a=4,-9$	M1 A1
		3
8(b)	$(\lambda-3)(\lambda-6)(\lambda+2)=\lambda^3-7\lambda^2+36=0$	B1
	$36\mathbf{I}=7\mathbf{A}^2-\mathbf{A}^3\Rightarrow 36\left(\mathbf{A}^{-1}\right)^2=7\mathbf{I}-\mathbf{A}$	M1
	$\left(\mathbf{A}^{-1}\right)^2=\frac{1}{36}\begin{pmatrix}4 & -1 & -1\\ 0 & 1 & 1\\ 0 & 0 & 9\end{pmatrix}$	M1 A1
		4

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8(c)	Eigenvalues of $\mathbf{A}$ are $3, 6$ and -2.	B1
	$\lambda=3: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	M1 A1
	$\lambda=6: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$ $\quad \lambda=-2: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 1 \\ 0 & 8 & -1 \end{vmatrix} = \begin{pmatrix} -9 \\ 5 \\ 40 \end{pmatrix}$	A1 A1
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 & -9 \\ 0 & 3 & 5 \\ 0 & 0 & 40 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 243 & 0 & 0 \\ 0 & 7776 & 0 \\ 0 & 0 & -32 \end{pmatrix}$ or $\mathbf{D} = \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 6^5 & 0 \\ 0 & 0 & -2^5 \end{pmatrix}$	M1 A1
		7

Q8.

3(a)	$\begin{vmatrix} 1 & -2 & -4 \\ 1 & -2 & k \\ -1 & 2 & 2 \end{vmatrix} = -4-2k+2(2+k)-4(0)=0$	M1 A1	Shows that determinant is zero.
		2	
3(b)	$x-2y-4z=1,$ $x-2y-4z=1, \quad \Rightarrow z=-1, x-2y=-3$ $-x+2y+2z=1,$ (or $-x+2y+2z=1$ is not parallel to $x-2y-4z=1$)	B1	Derives one equation with two unknowns or states that third plane is not parallel to the repeated one.
	Two of the planes are identical (or coincident).	B1	
	There is a line of intersection with the other plane.	B1	
		3	
3(c)	$x-2y-4z=1,$ $x-2y-2z=1, \quad \Rightarrow -1=1$ $-x+2y+2z=1,$	B1	Derives contradiction.
	Two parallel planes, not identical.	B1	
		2	
3(d)	$x-2y-4z=1,$ $x-2y+kz=1, \quad \Rightarrow (-4-k)z=0, -2z=2 \Rightarrow 0=2$ $-x+2y+2z=1,$	B1	Derives contradiction.
	The three planes form a triangular prism.	B1	
		2	